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# Yuansi Chen's result on the KLS conjecture

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**Resumen**

Estas notas constituyen un anexo del trabajo previo de los autores [AB2]. Aquí se incluye la demostración del reciente resultado de Yuansi Chen [Ch], en el que se mejoran sustancialmente los resultados incluidos en [AB2], y que fue simultaneo a su publicación. Se siguen las pautas de la exposición hecha por Bo'az Klartag en [K], pero manteniéndose dentro del esquema de demostración utilizado en el anterior trabajo [AB2].

**Abstract**

These notes constitute an annex to the previous work by the authors [AB2]. We include here the proof of the recent result by Yuansi Chen [Ch], in which the results included in [AB2] are substantially improved, and which was simultaneous to its publication. We follow the guidelines in the exposition made by Bo'az Klartag in [K], but remaining inside the scheme of the proof used in our previous work [AB2].

## 1 Introduction

The Kannan-Lovász-Simonovits (KLS) conjecture is a major open problem in asymptotic geometric analysis, which concerns Cheeger-type isoperimetric inequalities for log-concave probabilities, i.e., probabilities $\mu$ on $\mathbb{R}^n$ of the form $d\mu(x) = e^{-V(x)}dx$, with $V : \mathbb{R}^n \to (-\infty, \infty]$ a convex function. It was posed in [KLS] and can be stated in the following way:

**Conjecture 1.1** (KLS spectral gap conjecture). *There exists an absolute constant $C > 0$ such that, for any log-concave probability $\mu$ in $\mathbb{R}^n$*

$$(1) \qquad \mu^+(A) \geq \frac{C}{\sqrt{\|Cov_\mu\|_{op}}} \min\{\mu(A), \mu(A^c)\}, \qquad \textit{for any Borel set } A \subset \mathbb{R}^n$$

*where*

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

*being $A^\varepsilon = \{a + x : a \in A, |x| < \varepsilon\}$, and $\|Cov_\mu\|_{op}$ is the operator norm of the covariance matrix of $\mu$.*

Given a log-concave probability measure $\mu$ on $\mathbb{R}^n$, let us denote by $\psi_\mu$ the largest constant such that

$$\mu^+(A) \geq \frac{\psi_\mu}{\sqrt{\|\mathrm{Cov}_\mu\|_{op}}} \min\{\mu(A), \mu(A^c)\} \qquad \text{for any Borel set } A \subset \mathbb{R}^n.$$

Let us also denote by $\psi_n$, the infimum of the constants $\psi_\mu$ when $\mu$ runs over all log-concave probability measures on $\mathbb{R}^n$. That is,

$$\psi_n = \inf\{\psi_\mu \,:\, \mu \text{ is a log-concave probability on } \mathbb{R}^n\}.$$

Therefore, the KLS conjecture asks about the existence of a positive absolute constant $C > 0$ such that $\psi_n \geq C$ for every $n \in \mathbb{N}$. Let us point out that it is well known that there exists an absolute constant such that $\psi_\mu \leq C$ for every log-concave probability measure and that the KLS conjecture and can be reduced to the setting of isotropic log-concave probabilities, which are centered log-concave probabilities whose covariance matrix $\mathrm{Cov}_\mu$ is the identity matrix.

In a recent article [AB2] in this journal, the authors presented *Eldan's localization scheme* and proved, in a unified framework, the two best known estimates for $\psi_n$ which had been proved by Eldan [E] and Lee & Vempala [LV]. It can be stated in the following way:

**Theorem 1.1.** *Let $\psi_n$ be the best constant such that for any isotropic log-concave probability $\mu$ in $\mathbb{R}^n$ the following isoperimetric inequality holds*

$$\mu^+(A) \geq \psi_n \min\{\mu(A), \mu(A^c)\} \quad \textit{for any Borel set } A \subseteq \mathbb{R}^n.$$

*Then, there exists an absolute constant $C > 0$ such that*

$$\psi_n \geq \frac{C}{\min\{\sigma_n \log n, n^{1/4}\}},$$

*where* $\sigma_n = \sqrt{\sup \mathbb{E}_\mu \big||X| - \sqrt{n}\big|^2}$ *and the* sup *runs over all isotropic log-concave random vectors $X$ in $\mathbb{R}^n$.*

Simultaneously to the publication of [AB2], Yuansi Chen improved the best known estimate of $\psi_n$, by proving the following theorem:

**Theorem 1.2** (Y. Chen [Ch])**.** *There exists absolute constants $c_1, c_2 > 0$ such that for any isotropic log-concave probability $\mu$ in $\mathbb{R}^n$ the following isoperimetric inequality holds*

$$\mu^+(A) \geq c_1 \exp\big(-c_2\sqrt{\log n \cdot \log\log n}\big)\, \min\{\mu(A), \mu(A^c)\}$$

*for any Borel set $A \subseteq \mathbb{R}^n$.*

This paper tries to be an Appendix to the aforementioned paper [AB2], in which we include the proof of Chen's estimate in the framework developed there. The proof that we present here follows the original idea appearing in Eldan's seminal paper [E] (see also [E2] for an exposition of the technique). The same idea was also used by Lee & Vempala in their approach [LV] and also in our previous paper [AB2] . However the proof by [Ch] presents some formal differences. Namely, he preferred taking expectations in the isoperimetric inequalities rather than controlling how the measure of the individual 1/2-Borel sets evolves throughout Eldan's stochastical localization scheme. In order to get Chen's result we mimic the method used by Klartag [K], which uses a stopping time argument instead of the original reiteration method by Chen.

## 2 Preliminary results

We will follow the framework developed in [AB2], which we recall here in order to improve the readability of this annex. Nevertheless, we refer the reader to [AB2] for more detailed explanations.

*2.1 A first reduction*

We already mentioned in the introduction that one can just consider isotropic log-concave probability measures. It was also showed in [AB2] that it is enough to prove Conjecture 1.1 for isotropic log-concave measures with compact support. This condition ensured the existence and uniqueness of solution on the system of stochastic differential equations considered in the proof (see (2) below). One can reduce the class of isotropic log-concave probability measures to consider even further, by also assuming that their supports are contained in a Euclidean ball $r_n B_2^n$ of some large (but not "too large") radius $r_n$. We state it in the following lemma:

**Lemma 2.1.** *There exists an absolute constant* $C > 0$ *such that if for every isotropic log-concave probability measure* $\mu$ *with* $\operatorname{supp}\mu \subseteq Cn^5 B_2^n$ *we have that*

$$\psi_\mu \geq C_n,$$

*for some* $C_n > 0$*, then, we have that*

$$\psi_n \geq cC_n,$$

*where* $c$ *is an absolute constant.*

*Proof.* Let $d\mu(x) = e^{-V(x)}dx$ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let

$$d\mu_1 = \frac{e^{-V(x)}\chi_{n^5 B_2^n}(x)dx}{\int_{n^5 B_2^n} e^{-V(x)}dx}.$$

Notice that, by Paouris' inequality [BGVV, Theorem 5.2.1], there exists an absolute constant $c > 0$ such that

$$\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)}dx = \mu\{x \in \mathbb{R}^n \,:\, |x| \geq n^5\} \leq e^{-cn^5}.$$

Notice also that for any $\theta \in S^{n-1}$, by Hölder's inequality and by Borell's inequality, (see

[BGVV, Theorem 2.4.6])

$$\int_{\mathbb{R}^n\setminus n^5B_2^n}\langle x,\theta\rangle^2e^{-V(x)}dx\le\left(\int_{\mathbb{R}^n\setminus n^5B_2^n}e^{-V(x)}dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\langle x,\theta\rangle^4e^{-V(x)}dx\right)^{1/2}\le C_1e^{-cn^5}.$$

Therefore, for every $\theta\in S^{n-1}$

$$A\le 1-C_1e^{-cn^5}\le\mathbb{E}_{\mu_1}\langle X,\theta\rangle^2\le\frac{1}{1-e^{-cn^5}}\le B,$$

where $A,B$ are positive absolute constants and then $A\le\|\mathrm{Cov}_{\mu_1}\|_{op}\le B$. Thus, if we take $T\in GL(n)$ such that $d\mu_2(x)=d\mu_1(Tx)$, we have that supp $\mu_2\subseteq Cn^5B_2^n$, where $C$ is an absolute constant. By hypothesis, for this absolute constant $C$ we can ensure that $\psi_{\mu_2}\ge C_n$ and, consequently, $\psi_{\mu_1}\ge C_n$.

For every integrable locally Lipschitz function $g:\mathbb{R}^n\to\mathbb{R}$ such that $\mathbb{E}_\mu g=0$ we have that

$$\begin{aligned}
&|\mathbb{E}_{\mu_1}g|\int_{n^5B_2^n}e^{-V(x)}dx=|\mathbb{E}_{\mu_1}g-\mathbb{E}_\mu g|\int_{n^5B_2^n}e^{-V(x)}dx\\
&=\left|\left(\int_{n^5B_2^n}g(x)e^{-V(x)}\right)\left(1-\int_{n^5B_2^n}e^{-V(x)}dx\right)-\left(\int_{\mathbb{R}^n\setminus n^5B_2^n}g(x)e^{-V(x)}dx\right)\left(\int_{n^5B_2^n}e^{-V(x)}dx\right)\right|\\
&\le(\mathbb{E}_\mu g^2)^{1/2}\left[\left(\int_{n^5B_2^n}e^{-V(x)}dx\right)^{1/2}\left(\int_{\mathbb{R}^n\setminus n^5B_2^n}e^{-V(x)}\right)+\left(\int_{\mathbb{R}^n\setminus n^5B_2^n}e^{-V(x)}\right)^{1/2}\left(\int_{n^5B_2^n}e^{-V(x)}dx\right)\right]\\
&\le(\mathbb{E}_\mu g^2)^{1/2}\Big[e^{-cn^5}+e^{-\frac{c}{2}n^5}\Big]\le 2e^{-\frac{c}{2}n^5}(\mathbb{E}_\mu g^2)^{1/2}=2e^{-\frac{c}{2}n^5}(\mathrm{Var}_\mu g)^{1/2}.
\end{aligned}$$

and

$$\int_{\mathbb{R}^n\setminus n^5B_2^n}|g(x)|e^{-V(x)}dx\le\left(\int_{\mathbb{R}^n\setminus n^5B_2^n}e^{-V(x)}dx\right)^{1/2}(\mathrm{Var}_\mu g)^{1/2}\le e^{-cn^5}\left(\mathrm{Var}_\mu g\right)^{1/2}.$$

Therefore, taking into account the relation between Cheeger-type isoperimetric inequalities and Poincaré-type inequalities, and the equivalence between the constants in different Poincaré-type inequalities (see, for instance, [AB1, Theorem 1.11]), we have that for every integrable locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{E}_\mu g = 0$

$$\begin{aligned}
&\mathbb{E}_\mu|g| = \mathbb{E}_{\mu_1}|g| \int_{n^5 B_2^n} e^{-V(x)}dx + \int_{\mathbb{R}^n \setminus n^5 B_2^n} |g(x)|e^{-V(x)}dx \\
&\leq \mathbb{E}_{\mu_1}|g - \mathbb{E}_{\mu_1} g| \int_{n^5 B_2^n} e^{-V(x)}dx + |\mathbb{E}_{\mu_1} g| \int_{n^5 B_2^n} e^{-V(x)}dx + \int_{\mathbb{R}^n \setminus n^5 B_2^n} |g(x)|e^{-V(x)}dx \\
&\leq \frac{c_1}{\psi_{\mu_1}} \mathbb{E}_{\mu_1}|\nabla g| \int_{n^5 B_2^n} e^{-V(x)}dx + 2e^{-\frac{c}{2}n^5}(\mathrm{Var}_\mu g)^{1/2} + e^{-cn^5}\,(\mathrm{Var}_\mu g)^{1/2} \\
&\leq \frac{c_1}{\psi_{\mu_1}} \mathbb{E}_{\mu_1}|\nabla g| \int_{n^5 B_2^n} e^{-V(x)}dx + 3e^{-\frac{c}{2}n^5}(\mathrm{Var}_\mu g)^{1/2} \leq \frac{c_1}{\psi_{\mu_1}} \mathbb{E}_\mu|\nabla g| + \frac{c_2}{\psi_\mu} e^{-\frac{c}{2}n^5} \left(\mathbb{E}_\mu |\nabla g|^2\right)^{1/2}.
\end{aligned}$$

Since by Lee & Vempala's result and the fact that $\psi_\mu$ is bounded from above by an absolute constant, there exist absolute constants such that

$$\frac{c_2}{\psi_\mu} e^{-\frac{c}{2}n^5} \leq c_3 n^{1/4} e^{-\frac{c}{2}n^5} \leq c_4 \leq \frac{c_5}{\psi_{\mu_1}}$$

and then, for every integrable locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{E}_\mu g = 0$,

$$\mathbb{E}_\mu|g| \leq \frac{c_6}{\psi_{\mu_1}} \||\nabla g|\|_\infty.$$

Therefore, since for every integrable locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ we have that $g_1 = g - \mathbb{E}_\mu g$ verifies that $\mathbb{E}_\mu g_1 = 0$ and $\nabla g_1 = \nabla g$, we have that for every integrable locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$

$$\mathbb{E}_\mu|g - \mathbb{E}_\mu g| \leq \frac{c_6}{\psi_{\mu_1}} \||\nabla g|\|_\infty.$$

Thus,

$$\psi_\mu \geq c_7 \psi_{\mu_1} \geq c_7 C_n.$$

□

**Remark 2.2.** *Let us point out that the same proof would work with any power of $n$ larger than $\frac{1}{2}$ instead of $n^5$.*

From now on, we will consider $d\mu(x) = f(x)dx$ to be an isotropic log-concave probability measure such that $\operatorname{supp}\mu \subseteq Cn^5 B_2^n$ where $C$ is the absolute constant in Lemma 2.1. Let us recall that we consider Lee & Vempala's choice in the system of stochastic differential equations in Eldan's localization scheme. That is,

$$dc_t = b_t dt + dW_t, \qquad c_0 = 0 \tag{2}$$

where $W_t$ a $n$-dimensional Wiener process and $b_t$ is the barycenter of the density $f_t(x)$ given by

$$f_t(x) = \frac{e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x)dx} \qquad b_t = \int_{\mathbb{R}^n} x f_t(x)dx. \tag{3}$$

The probability measure with density $f_t(x)$ will be denoted by $\mu_t$ and its covariance matrix will be denoted by $A_t$.

Let us recall that, given $\mu$ an isotropic log-concave probability, our goal is to find two values $\Theta, C > 0$ such that for any Borel set $E \subseteq \mathbb{R}^n$ with $\mu(E) = 1/2$.

$$\mu(E^\Theta \setminus E) \geq C,$$

where $E^\Theta = \{e + x \in \mathbb{R}^n : e \in E, |x| < \Theta\}$ is the $\Theta$-dilation of $E$, in order to apply the following proposition, which can be found on [AB2, Proposition 2.3]:

**Proposition 2.3** ([M])**.** *Let $\mu$ be a log-concave probability on $\mathbb{R}^n$. Assume that there exist two positive numbers $\Theta, C > 0$ such that*

$$\mu(E^\Theta \setminus E) \geq C$$

*for any Borel set $E \in \mathbb{R}^n$ such that $\mu(E) = \frac{1}{2}$. Then,*

$$\mu^+(A) \geq \frac{C}{\Theta} min\{\mu(A), \mu(A^c)\} \qquad \textit{for any Borel set } A \subset \mathbb{R}^n.$$

In order to control the probability of dilations of Borel sets, the following concentration results for *more log-concave than Gaussian* probabilities (see [AB2, Proposition 2.4]) can be applied

**Proposition 2.4.** *Let $\phi$ be a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and let $t > 0$. Assume that*

$$d\mu(x) = e^{-\phi(x) - \frac{t}{2}|x|^2} dx,$$

*is a centered probability on $\mathbb{R}^n$. Then for every Borel set $A \subset \mathbb{R}^n$ such that*

$$\frac{1}{10} \leq \mu(A) \leq \frac{9}{10}$$

*we have*

$$\mu\left(A^{\frac{D}{\sqrt{t}}}\right) \geq \frac{95}{100},$$

*where $D > 0$ is a suitably chosen absolute constant independent of every other parameter and $A^{D/\sqrt{t}}$ is the $D/\sqrt{t}$-dilation of $A$.*

**Remark 2.5.** *Notice that, due to the fact that*

$$d\mu_t(x) = \frac{e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x) dx}{\int_{\mathbb{R}^n} e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x) dx}$$

*is more log-concave than the Gaussian probability, as a trivial application of both propositions,*

$$\mu_t^+(A) \geq c_1 \sqrt{t} \min\{\mu_t(A), \mu_t(A^c)\}, \qquad \text{for any Borel set } A \subset \mathbb{R}^n,$$

*where, a fortiori, $0 < c_1 < 1$ is an absolute constant.*

In the sequel $E$ will denote a fixed Borel set in $\mathbb{R}^n$ such that $\mu(E) = 1/2$. We introduce the stochastic process

$$g_E(t) = g(t) = \mu_t(E) = \int_E f_t(x) dx, \qquad t \geq 0,$$

where $\mu_t$ and $f_t(x)$ are defined by the system of stochastic differential equations (2) and by (3). It is obvious that $g(0) = 1/2$, $\forall \omega \in \Omega$, since $f_0(x) = f(x)$ for every $\omega \in \Omega$. Besides, $(g(t))_{t \geq 0}$ is a martingale (see [AB2, Section 4]) and for every $t \geq 0$ the expected value of $g(t)$ is $\mathbb{E}_{\mathbb{P}} g(t) = 1/2$.

Let $T > 0$ be a time to be precised later and notice that for any $\Theta > 0$, since also

$(g_{E^\Theta\setminus E}(t))_{t\geq 0}$ is a martingale,

$$\mu(E^\Theta \setminus E) = \int_{E^\Theta\setminus E} f(x)dx = \int_{E^\Theta\setminus E} \mathbb{E}_{\mathbb{P}} f_T(x)dx$$
$$= \mathbb{E}_{\mathbb{P}} \int_{E^\Theta\setminus E} f_T(x)dx = \mathbb{E}_{\mathbb{P}}\mu_T(E^\Theta \setminus E).$$

In order to apply the preceding propositions we will consider the event

$$\mathcal{G} = \{\omega \in \Omega : |g(T) - 1/2| \leq 1/4\}.$$

By Proposition 2.4 and the way that the densities $f_t$ are defined, we will have that there exists some absolute constant $D > 0$ such that for $\omega \in \mathcal{G}$ we will have $\mu_T(E^{D/\sqrt{T}}) \geq 0.95$ and therefore, by Markov's inequality,

$$\mu(E^{D/\sqrt{T}} \setminus E) = \mathbb{E}_{\mathbb{P}}\mu_T(E^{D/\sqrt{T}} \setminus E) \geq (0.95 - 0.5)\mathbb{P}(\mathcal{G}) = \frac{9}{20}\,\mathbb{P}(\mathcal{G}).$$

Hence, if we find $T, C_1 > 0$ independent of $E$ such that $\mathbb{P}(\mathcal{G}) > C_1$ then we will get that

$$\mu^+(A) \geq \frac{C_1\sqrt{T}}{D}\min\{\mu(A), \mu(A^c)\} \qquad \forall A \text{ Borel set } \subset \mathbb{R}^n.$$

### *2.3 Estimating the probability of $\mathcal{G}$*

In order to bound from below $\mathbb{P}\{|g(T) - 1/2| \leq 1/4\}$ by a positive absolute constant, for a particular choice of a time $T$, we recall that, as obtained in [AB2, Section 4],

$$g(T) - \frac{1}{2} = g(T) - g(0) = \int_0^T dg(t) = \int_0^T \langle \eta_t, dW_t\rangle$$

where $\eta_t = \displaystyle\int_E f_t(x)(x - b_t)dx$, being $f_t$ the density of the probability measure $\mu_t$ defined in (3).

The function $g(t)$ is a martingale and so, by Dambis, Dubins-Schwarz theorem, [AB2, Proposition 2.7], we have that in distribution

$$g(T) - g(0) = \bar{W}_{[g]_T},$$

where $\bar{W}_s$ is a Wiener process and $[g]_T$ is the quadratic variation of $g$, which is,

$$[g]_T = \int_0^T |\eta_t|^2 dt.$$

Hence, for any $M > 0$,

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} = \mathbb{P}\{|\bar{W}_{[g]_T}| > 1/4\} \leq \mathbb{P}\{[g]_T > M\} + \mathbb{P}\left\{\max_{0\leq t\leq M} |\bar{W}_t| > \frac{1}{4}\right\}.$$

We will bound both summands from above for an appropriate choice of $M$. Taking into account that for every $t \geq 0$

$$\begin{aligned}|\eta_t| = \left\langle \eta_t, \frac{\eta_t}{|\eta_t|}\right\rangle &= \int_E f_t(x) \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|}\right\rangle dx \\ &\leq \sqrt{\mathbb{E}_{\mu_t} \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|}\right\rangle^2} = \sqrt{\left\langle A_t \frac{\eta_t}{|\eta_t|}, \frac{\eta_t}{|\eta_t|}\right\rangle} \leq \sqrt{\|A_t\|_{\mathrm{op}}},\end{aligned}$$

we have that

$$[g]_T \leq \int_0^T \|A_t\|_{\mathrm{op}}\, dt$$

and then

$$\mathbb{P}\{[g]_T > M\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\mathrm{op}}\, dt > M\right\}. \tag{4}$$

On the other hand, $\left(-\bar{W}_t\right)_{t\geq 0}$ is also a Brownian motion and then, by the reflection principle, [AB2, Proposition 2.6], we have

$$\begin{aligned}\mathbb{P}\left\{\max_{0\leq t\leq M} |\bar{W}_t| > \frac{1}{4}\right\} &\leq \mathbb{P}\left\{\max_{0\leq t\leq M} \bar{W}_t > \frac{1}{4}\right\} + \mathbb{P}\left\{\max_{0\leq t\leq M} -\bar{W}_t > \frac{1}{4}\right\} \\ &= 4\mathbb{P}\left\{\bar{W}_M > \frac{1}{4}\right\} \leq 4\exp\left(-\frac{1}{32M}\right).\end{aligned} \tag{5}$$

Hence

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\mathrm{op}}\, dt > M\right\} + 4\exp\left(-\frac{1}{32M}\right).$$

Our purpose is to find a suitable $M > 0$ and $T > 0$ such that we can ensure that the latter upper bound on the probability of $\mathcal{G}^c$ is strictly smaller than 1. We will take $M = 1/256$. Then

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\mathrm{op}}\, dt > \frac{1}{256}\right\} + 4\exp(-8)$$

and, by showing that for an appropriate choice of $T$ the latter probability is bounded above by $\frac{1}{10}$, we will obtain the desired lower bound on the probability of $\mathcal{G}$.

## 3 Chen's estimate on $\|A_t\|_{op}$

Let us recall that the main result, which allowed to obtain Theorem 1.1 by following the described strategy, was the following:

**Proposition 3.1.** *[AB2, Proposition 5.1] Given the system of stochastic differential equations* (2)*, let $A_t$ be the covariance matrix of the measure $\mu_t$ defined by* (3)*. Let $p \geq 2$ be an integer. Then*

$$d(\mathit{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t\rangle$$

*where $\delta_t$ is an adapted, with bounded variation process, such that*

$$\delta_t \leq \begin{cases} C\, p^2\sigma_n^2 \log n\, \mathit{Tr}(A_t^p)^{1+\frac{1}{p}}, & \text{if } p \geq 3 \\ C\,\mathit{Tr}(A_t^2)^{3/2}, & \text{if } p = 2 \end{cases}$$

*and*

$$|v_t| \leq Cp\,\mathit{Tr}(A_t^p)^{1+\frac{1}{2p}} \qquad \forall p \geq 2,$$

*where $C > 0$ is an absolute constant and $\sigma_n^2 = \sup \mathbb{E}\big||X| - \sqrt{n}\big|^2$ and the* sup *runs over all isotropic log-concave random vectors in $\mathbb{R}^n$.*

In this section we present Chen's improvement on Proposition 3.1 and some consequences on the estimates of $\|A_t\|_{op}$.

**Proposition 3.2.** *Given the system of stochastic differential equations* (2)*, let $A_t$ be the covariance matrix of the measure $\mu_t$ defined by* (3)*. Let $p \geq 3$ be an integer. Then*

$$d(\mathit{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t\rangle$$

*where $\delta_t$ is an adapted, with bounded variation process, such that*

$$\delta_t \leq Cp^2\, Tr(A_t^p) \min\left\{\frac{1}{t}, \frac{\|A_t\|_{op}}{\psi_n^2}\right\}, \qquad |v_t| \leq Cp\, Tr(A_t^p)^{1+\frac{1}{2p}},$$

*where $C > 0$ is an absolute constant and $\psi_n$ is defined as in Theorem 1.1.*

*Proof.* The estimate for $|v_t|$ is the same appearing in Proposition 3.1. We will follow the ideas appearing in [K] in order to estimate $\delta_t$. Let us recall that, after Lee & Vempala's result collected in Theorem 1.1, we know that $\psi_n \geq \frac{C}{n^{1/4}}$ for some absolute constant $C$ and that one trivially has $\psi_n \leq C$ for another absolute constant $C$.

According to the proof of Proposition 3.1,

$$\begin{aligned}
\delta_t &\leq \frac{1}{2}p(p-1)\sum_{i=1}^n (\alpha_{ii})^{p-2}|\xi_{ii}|^2 + \sum_{\substack{i\neq j\\ 0\leq k\leq p-2}} (\alpha_{ii})^k(\alpha_{jj})^{p-k-2}|\xi_{ij}|^2\\
&\leq \frac{1}{2}p(p-1)\sum_{i=1}^n (\alpha_{ii})^{p-2}|\xi_{ii}|^2 + 2\sum_{\substack{i\neq j\\ 0\leq k\leq p-2}} (\alpha_{ii})^{p-2}|\xi_{ij}|^2\\
&\leq p(p-1)\sum_{i=1}^n (\alpha_{ii})^{p-2}|\xi_{ii}|^2 + p(p-1)\sum_{i<j}(\alpha_{ii})^{p-2}|\xi_{ij}|^2\\
&\leq p(p-1)\sum_{i,j=1}^n (\alpha_{ii})^{p-2}|\xi_{ij}|^2,
\end{aligned}$$

where $(v_i)_{i=1}^n$ is an orthonormal basis of eigenvectors of the covariance matrix $A_t$, $\alpha_{ij} = \alpha_{ij}(t) = \langle A_t v_i, v_j\rangle$, ordered in such a way that $\alpha_{11} \geq \alpha_{22} \geq \cdots \geq \alpha_{nn}$, and $\xi_{ij}$ are the vectors $\xi_{ij} = \xi_{i,j}(t) = \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\langle x-b_t, v_j\rangle(x-b_t) \in \mathbb{R}^n$.

Let, for any $1 \leq i \leq n$, $\xi_i$ be the symmetric matrix

$$\xi_i = \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle(x-b_t)\otimes(x-b_t).$$

Then,

$$\mathrm{Tr}(\xi_i^2) = \sum_{j,k=1}^n \left(\mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\langle x-b_t, v_j\rangle\langle x-b_t, v_k\rangle\right)^2 = \sum_{j=1}^n |\xi_{ij}|^2.$$

Therefore,

$$\delta_t \leq p(p-1)\sum_{i=1}^{n}(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2).$$

Furthermore, since

$$\xi_i^2 = \xi_i \circ \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle(x-b_t)\otimes(x-b_t) = \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle(x-b_t)\otimes\xi_i(x-b_t)$$

and, for every $1 \leq i \leq n$, $\mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle = 0$ we have that

$$\begin{aligned}
\mathrm{Tr}(\xi_i^2) &= \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\langle x-b_t, \xi_i(x-b_t)\rangle \\
&= \mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\left(\langle x-b_t, \xi_i(x-b_t)\rangle - \mathbb{E}_{\mu_t}\langle y-b_t, \xi_i(y-b_t)\rangle\right) \\
&\leq \text{ (by Cauchy-Schwarz inequality)} \\
&\leq \sqrt{\alpha_{ii}}\sqrt{\mathrm{Var}_{\mu_t}\langle x-b_t, \xi_i(x-b_t)\rangle}.
\end{aligned}$$

Taking into account the well-known relationship of Cheeger-type isoperimetric inequalities with Poincare's inequality (see, for instance, [AB1, Theorems 1.1 and 1.8]) we have that for any locally Lipschitz integrable function $g$ and any log-concave probability $\mu$ in $\mathbb{R}^n$

$$\mathrm{Var}_\mu g \leq \frac{C\|\mathrm{Cov}_\mu\|_{\mathrm{op}}}{\psi_n^2}\mathbb{E}_\mu|\nabla g|^2$$

for some absolute constant $C > 0$. Besides, by Remark 2.5, for any locally Lipschitz integrable function $g$

$$\mathrm{Var}_{\mu_t}(g) \leq \frac{C}{t}\mathbb{E}_{\mu_t}|\nabla g|^2.$$

Therefore,

$$\mathrm{Var}_{\mu_t}\langle x-b_t, \xi_i(x-b_t)\rangle \leq C\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}\mathbb{E}_{\mu_t}\left|\nabla\langle x-b_t, \xi_i(x-b_t)\rangle\right|^2.$$

Since

$$\nabla\langle x-b_t, \xi_i(x-b_t)\rangle = (\xi_i + \xi_i^t)(x-b_t) = 2\xi_i(x-b_t),$$

we have that

$$\mathrm{Tr}(\xi_i^2) \le C\sqrt{\alpha_{ii}}\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathbb{E}_{\mu_t}|\xi_i(x-b_t)|^2}$$
$$\le C\sqrt{\alpha_{ii}}\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathrm{Tr}(A_t\xi_i^2)}.$$

Hence,

$$\begin{aligned}\sum_{i=1}^n(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2) &\le C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sum_{i=1}^n \alpha_{ii}^{p-3/2}\sqrt{\mathrm{Tr}(A_t\xi_i^2)}\\ &\le \text{ (by Cauchy-Schwarz inequality)}\\ &\le C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\sum_{i=1}^n \alpha_{ii}^p}\sqrt{\sum_{i=1}^n \alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2)}\\ &= C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathrm{Tr}(A_t^p)}\sqrt{\sum_{i=1}^n \alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2)}.\end{aligned}$$

Now, since the matrix $A_t$ is diagonal in the basis $(v_j)_{j=1}^n$ and the entry $(\xi_i^2)_{jj}$ of the matrix $\xi_i^2$ is

$$(\xi_i^2)_{jj} = \sum_{k=1}^n \left(\mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\langle x-b_t, v_j\rangle\langle x-b_t, v_k\rangle\right)^2 = |\xi_{ij}|^2,$$

we have that

$$\sum_{i=1}^n \alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2) = \sum_{i,j=1}^n \alpha_{ii}^{p-3}\alpha_{jj}(\xi_i^2)_{jj} = \sum_{i,j=1}^n \alpha_{ii}^{p-3}\alpha_{jj}|\xi_{ij}|^2.$$

For every $1 \le i, j \le n$ we have that $\alpha_{ii}^{p-3}\alpha_{jj} \le \max\{\alpha_{ii}^{p-2}, \alpha_{jj}^{p-2}\} \le \alpha_{ii}^{p-2} + \alpha_{jj}^{p-2}$. Thus,

$$\sum_{i=1}^n \alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2) \le \sum_{i,j=1}^n \alpha_{ii}^{p-2}|\xi_{ij}|^2 + \sum_{i,j=1}^n \alpha_{jj}^{p-2}|\xi_{ij}|^2 = 2\sum_{i,j=1}^n \alpha_{ii}^{p-2}|\xi_{ij}|^2 = 2\sum_{i=1}^n \alpha_{ii}^{p-2}\mathrm{Tr}(\xi_i^2)$$

and then

$$\sum_{i=1}^{n}(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2)\leq C\sqrt{\min\left\{\frac{1}{t},\frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathrm{Tr}(A_t^p)}\sqrt{\sum_{i=1}^{n}\alpha_{ii}^{p-2}\mathrm{Tr}(\xi_i^2)},$$

where $C$ is an absolute constant. Therefore,

$$\sum_{i=1}^{n}(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2)\leq C\min\left\{\frac{1}{t},\frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}\mathrm{Tr}(A_t^p)$$

and

$$\delta_t\leq p(p-1)\sum_{i=1}^{n}(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2)\leq Cp^2\min\left\{\frac{1}{t},\frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}\mathrm{Tr}(A_t^p),$$

where $C$ is an absolute constant. □

**Proposition 3.3.** *There exists an absolute constant $c>0$ and $n_0\in\mathbb{N}$ such that if $T_0:=\frac{c\psi_n^2}{\log n}$, then for every $p\geq 3$ and every $n\geq n_0$, we have*

$$\max_{t\in[0,T_0]}\mathbb{E}_{\mathbb{P}}\|A_t\|_{op}\leq 3\qquad \textit{and}\qquad \max_{t\in[0,T_0]}\mathbb{E}_{\mathbb{P}}(\mathit{Tr}(A_t^p))^{1/p}\leq 3n^{1/p}.$$

*Proof.* First of all notice that, since $\psi_n$ is bounded from above by an absolute constant, we can choose an absolute $0<c_1<1/2$ and take $T_0:=\frac{c_1\psi_n^2}{40^2C\log n}$, where $C$ is the constant appearing in Proposition 3.2.

Let us consider the stopping time $\tau(\omega)=\inf\big\{t>0;\|A_t\|_{\mathrm{op}}\geq 2\big\}$. It is clear that, by continuity, for $\omega\in\Omega$ and $t\leq\tau(\omega)$, $\|A_t\|_{\mathrm{op}}\leq 2$. We define the following stochastic process $X_t(\omega)=\mathrm{Tr}(A^p_{\min\{t,\tau(\omega)\}})$. It is an Itô process. Indeed, $dX_t=\bar{\delta}_t dt+\langle\bar{v}_t,dW_t\rangle$ where

$$\bar{\delta}_t=\begin{cases}\delta_t & \text{if } t<\tau(\omega)\\ 0, & \text{otherwise}\end{cases}\qquad\text{and}\quad \bar{v}_t=\begin{cases}v_t & \text{if } t<\tau(\omega)\\ 0, & \text{otherwise .}\end{cases}$$

If $0\leq t<\tau(\omega)$,

$$\bar{\delta}_t=\delta_t\leq Cp^2\frac{2}{\psi_n^2}\mathrm{Tr}(A_t^p)=Cp^2\frac{2}{\psi_n^2}X_t.$$

Since the latter inequality is trivially true if $t\geq\tau(\omega)$ we have that it holds for every $t>0$.

It is clear that the stochastic process given by $Z_t=\int_0^t\langle\bar{v}_s,dW_s\rangle$ is a martingale and then $\mathbb{E}_{\mathbb{P}} Z_t = 0$. Hence the deterministic function $\mathbb{E}_{\mathbb{P}} X_t$ verifies

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}} X_t \leq Cp^2 \frac{2}{\psi_n^2}\mathbb{E}_{\mathbb{P}} X_t, \qquad \text{a.e. } t > 0.$$

Then, taking into account that $\mathbb{E}_{\mathbb{P}} X_0 = n$ and that $0 < c_1 < \frac{1}{2}$, we have that for every $t \in [0, T_0]$

$$\mathbb{E}_{\mathbb{P}} X_t \leq n \exp\left(\frac{2Cp^2}{\psi_n^2} T_0\right) \leq n \exp\left(\frac{p^2}{40^2 \log n}\right)$$

Optimizing for $p = [40 \log n]$ we obtain

$$\mathbb{E}_{\mathbb{P}} X_t \leq n \exp(\log n) = n^2 \qquad \forall t \in [0, T_0].$$

Therefore, since for this value of $p$ and $n \geq 3$

$$X_{\tau(\omega)} = \mathrm{Tr}(A_{\tau(\omega)}^{[40 \log n]}) \geq \|A_{\tau(\omega)}\|_{\mathrm{op}}^{30 \log n} = 2^{30 \log n}, \qquad \forall t \in [0, T_0]$$

we achieve, using Markov's inequality, that for every $t \in [0, T_0]$

$$n^2 \geq \mathbb{E}_{\mathbb{P}} X_t \geq \int_{\{\omega :\, t > \tau(\omega)\}} X_t d\mathbb{P} = \int_{\{\omega :\, t > \tau(\omega)\}} X_{\tau(\omega)} d\mathbb{P} \geq 2^{30 \log n} \mathbb{P}\big\{\omega \,:\, t > \tau(\omega)\big\}$$

and then

$$\mathbb{P}\big\{\omega \,:\, t > \tau(\omega)\big\} \leq \frac{n^2}{n^{30 \log 2}}, \qquad \forall t \in [0, T_0].$$

Then, for every $n \geq n_0$, for some $n_0 \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{P}} \|A_t\|_{\mathrm{op}} \leq C_1 \frac{n^{12}}{n^{30 \log 2}} + 2 \leq 3 \qquad \forall t \in [0, T_0],$$

where we have used that, since we are assuming (as mentioned in Section 2.2) that $\mathrm{supp}\mu \subseteq Cn^5 B_2^n$,

$$\|A_t\|_{\mathrm{op}} = \sup_\theta \mathbb{E}_{\mu_t} \langle x - b_t, \theta\rangle^2 \leq 4\big(\mathrm{diameter}(\mathrm{supp}\ \mu_t)\big)^2 = 4\big(\mathrm{diameter}(\mathrm{supp}\ \mu)\big)^2 \leq C_1 n^{10}.$$

Besides, since for any $p \geq 3$,

$$\|A_t\|_{\mathrm{op}} \leq \big(\mathrm{Tr}(A_t^p)\big)^{1/p} \leq \|A_t\|_{\mathrm{op}} n^{1/p}$$

we have that

$$\mathbb{E}_{\mathbb{P}}\big(\mathrm{Tr}(A_t^p)\big)^{1/p} \leq 3n^{1/p} \qquad \forall t \in [0, T_0].$$

$\square$

**Lemma 3.4.** *Let* $T_0 = c\dfrac{\psi_n^2}{\log n}$ *as in Proposition 3.3. Let* $T_1, T_2 > 0$ *be such that* $0 < T_1 \leq T_0$ *and* $0 < T_1 \leq T_2$*. Then, for every* $p \geq 3$ *and* $n \geq n_0$*, for some* $n_0 \in \mathbb{N}$*, we have that*

$$\mathbb{P}\left\{\int_0^{T_1} \|A_t\|_{op} dt \geq \frac{1}{512}\right\} \leq 1536\, T_1$$

*and*

$$\mathbb{P}\left\{\int_{T_1}^{T_2} \|A_t\|_{op} dt \geq \frac{1}{512}\right\} \leq 1536\, n^{1/p} \left(\frac{T_2}{T_1}\right)^{Cp} T_2.$$

*Proof.* By Markov's inequality, since $T_1 < T_0$ and, as seen in Proposition 3.3, $\mathbb{E}_{\mathbb{P}}\|A_t\|_{\mathrm{op}} \leq 3$ for every $t \in [0, T_0]$,

$$\mathbb{P}\left\{\int_0^{T_1} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{512}\right\} \leq 512\mathbb{E}_{\mathbb{P}} \int_0^{T_1} \|A_t\|_{\mathrm{op}} dt = 512 \int_0^{T_1} \mathbb{E}_{\mathbb{P}}\|A_t\|_{\mathrm{op}} dt \leq 1536 T_1.$$

Consider the stochastic process $H_t = \big(\mathrm{Tr}(A_t^p)\big)^{1/p}$ which is an Itô process. Therefore, by Itô's formula,

$$dH_t = \frac{1}{p}\big(\mathrm{Tr}(A_t^p)\big)^{1/p-1} d\big(\mathrm{Tr}(A_t^p)\big) + \frac{1}{2p}\left(\frac{1}{p} - 1\right)\big(\mathrm{Tr}(A_t^p)\big)^{1/p-2} d[\mathrm{Tr}(A_t^p)]_t = \eta_t dt + dM_t,$$

where, by Proposition 3.2, $dM_t$ is a martingale with $M_0 = 0$ and $\eta_t$ is an adapted process such that $\eta_t \leq C\frac{p}{t} H_t$. Taking expectation we have

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}} H_t \leq C\,\frac{p}{t}\,\mathbb{E}_{\mathbb{P}} H_t \Leftrightarrow \frac{d}{dt}\log\left(\mathbb{E}_{\mathbb{P}} H_t\right) \leq C\,\frac{p}{t}.$$

Integrating in the interval $[T_1, t] \subseteq [T_1, T_2]$ we deduce that for any $T_1 < t < T_2$

$$\mathbb{E}_{\mathbb{P}} H_t \leq \mathbb{E}_{\mathbb{P}} H_{T_1} \left(\frac{t}{T_1}\right)^{Cp}.$$

Taking into account that, by Proposition 3.3, $\mathbb{E}_{\mathbb{P}} H_{T_1} \leq 3n^{1/p}$, for any value of $p \geq 3$, we obtain

$$\mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} \|A_t\|_{\mathrm{op}} dt \leq \mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} H_t dt = \int_{T_1}^{T_2} \mathbb{E}_{\mathbb{P}} H_t dt \leq 3n^{1/p} \int_{T_1}^{T_2} \left(\frac{t}{T_1}\right)^{Cp} dt \leq 3n^{1/p} \left(\frac{T_2}{T_1}\right)^{Cp} T_2.$$

Hence, by Markov's inequality,

$$\mathbb{P}\left\{\int_{T_0}^{T_1} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{512}\right\} \leq 512\, \mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} \|A_t\|_{\mathrm{op}} dt \leq 1536 n^{1/p} \left(\frac{T_2}{T_1}\right)^{Cp} T_2.$$

□

## 4 Proof of Chen's Theorem

In this section we complete the proof of Chen's estimate of $\psi_n$.

*Proof of Theorem 1.2.* Let $T_0 = c\dfrac{\psi_n^2}{\log n}$ for some absolute constant $c$, as in Proposition 3.3 and let $T_1 = T_0$, $p = \sqrt{\dfrac{\log n}{\log\log n}}$, and $T_2 = \alpha\, n^{-\frac{1}{p(Cp+1)}} T_0^{\frac{Cp}{Cp+1}}$ for this fixed value of $p$ and some $0 < \alpha < 1$ to be precised later.

There are two possibilities: either $T_2 \leq T_0$ or $T_2 > T_0$. Assume first that $T_2 \leq T_0$. According to Lemma 3.4, and taking into account that $\psi_n \leq C$ for some positive absolute constant $C$, we have that

$$\begin{aligned}\mathbb{P}\left\{\int_0^{T_2} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{256}\right\} &\leq \mathbb{P}\left\{\int_0^{T_2} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{512}\right\} \leq 1536\, T_2 \\ &\leq 1536\, T_0 \leq \frac{1536 c \psi_n^2}{\log n} \leq \frac{1}{10}\end{aligned}$$

for every $n > n_0$, for some $n_0 \in \mathbb{N}$.

Assume now that $T_2 > T_0$. By Lemma 3.4, and taking into account that $1 + x \leq 2x$

whenever $x \geq 1$, we have

$$\mathbb{P}\left\{\int_0^{T_2} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{256}\right\} \leq \mathbb{P}\left\{\int_0^{T_0} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{512}\right\} + \mathbb{P}\left\{\int_{T_0}^{T_2} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{512}\right\}$$
$$\leq 1536 \left(T_0 + n^{1/p} \left(\frac{T_2}{T_0}\right)^{Cp} T_2, \right) = 1536 T_0 \left(1 + n^{1/p} \left(\frac{T_2}{T_0}\right)^{Cp+1}\right)$$
$$\leq 3072\, T_0\, n^{1/p} \left(\frac{T_2}{T_0}\right)^{Cp+1} = 3072\, \alpha^{Cp+1} < \frac{1}{10}$$

whenever $n > n_1$, for some $n_1 \in \mathbb{N}$, by choosing $0 < \alpha < 1$ small enough.

Therefore, we can fix $0 < \alpha < 1$ such that there exists $n_0$ such that if $n \geq n_0$

$$\mathbb{P}\left\{\int_0^{T_2} \|A_t\|_{\mathrm{op}} dt \geq \frac{1}{256}\right\} \leq \frac{1}{10}.$$

By the arguments exposed in Section 2, we have that there exists an absolute constant $c_1 > 0$ such that

$$\psi_\mu \geq c_1 \sqrt{T_2} \geq c_2 \sqrt{\alpha} n^{-\frac{1}{2p(Cp+1)}} T_0^{\frac{Cp}{2(Cp+1)}} = c_3 \sqrt{\alpha} n^{-\frac{1}{2p(Cp+1)}} \frac{\psi_n^{\frac{Cp}{Cp+1}}}{(\log n)^{\frac{Cp}{2(Cp+1)}}}.$$

Since this inequality is true for any isotropic log-concave probability whose support is contained in $C_1 n^5 B_2^n$, where $C_1$ is the absolute constant in Lemma 2.1, we obtain that

$$\psi_n \geq c_4 \sqrt{\alpha} n^{-\frac{1}{2p(Cp+1)}} \frac{\psi_n^{\frac{Cp}{Cp+1}}}{(\log n)^{\frac{Cp}{2(Cp+1)}}}$$

and then

$$\psi_n \geq \left(c_4 \sqrt{\alpha}\right)^{Cp+1} \frac{n^{-\frac{1}{2p}}}{(\log n)^{\frac{Cp}{2}}} = c_5 \exp\left( - c_6 \sqrt{\log n \cdot \log\log n}\right),$$

which finishes the proof. □

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