Rev. Real Academia de Ciencias. Zaragoza. **66:** 63–104, (2011).
ISSN: 0370-3207

Groups of order less than 32, revisited

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Abstract

We consider all finite groups G up to order 32 (there are 93 of them) from a different point of view as usually seen. We consider first possible values for the number r of classes of conjugate elements from the relations $N \equiv |G| = \sum_{i=1}^{r} d_i^2$, where d_i is the dimension of the i-th irreducible complex representation, and $|G| = \sum_{n=1}^{r} n \cdot m_s$, the partition of elements in classes. For each possible value of N, we extract the maximum information: abelianized group, center, and some details on the classes of (outer) automorphisms. In several cases this is enough to characterize some of these groups completely.

MSC Classification: 20B05, 20D45

1 Introduction

Today finite groups up to order 2000 have been classified [5]; also, extensive studies do exist for groups e.g. up to order 32, or order power of a prime; see, for example, Coxeter-Moser, [10]; Thomas-Wood, [29]; Hall [18]; Berkovitz [3], [4] and Huppert [20]; for elementary details, up to order 8 see e.g. Lederman [22]; for order 16, Wild [32], etc. In this review we study these groups again, up to $|G| < 32$ (there are 48 abelian for a total of $48 + 45 = 93$ distinct groups), with a different point of view: our starting point will be the orthogonality*Burnside relation*, relating the order $|G|$ of group G with the dimensions d_i of the r inequivalent irreducible complex representations (*irreps*), and with the same number r of classes of conjugate elements, namely the two relations

$$|G| = \sum_{i=1}^{r} d_i^2 = 1 \cdot 1^2 + (N-1) \cdot 1^2 + M \cdot d_2^2 + \cdots , \tag{1.1}$$

$$|G| = \sum_{n=1}^{r} n \cdot m_s = 1 \cdot 1_1 + \sum_{n} n \cdot 1_{s\neq 1} + \cdots , \tag{1.2}$$

where $1, 2, \cdots i, \cdots r$ label the *irreps*, of dimension d_i, with also the r classes, expressed in the second equation, where $n \cdot m_s$ means there are n classes of conjugate elements, of m elements each, of period (order) s; we stress that there are $N \geq 1$ one-dim *irreps*, with the identical always, and the identity e is the first summand $1 \cdot 1_1$ in (1.2). The information enclosed in (1.1) and (1.2) is plentiful: number of classes r, order of the abelianized group $|G_{ab}| = N = 1 + (N-1)$, where $G_{ab} = G/G'$ and G' being the derived or commutator (sub-)group; order of the center, $|Z(G)| = 1 + \sum_n n \cdot 1_{s\neq 1}$, etc. *Automorphisms* can be detected as repeated elements of same period in the abelian case; in non-abelian groups, with $1 \leq N < |G|$, types of *outer* automorphisms are delated by repeated classes, etc. Of course, for each relation (1.1) and (1.2) there might be several groups.

We shall employ the usual definitions and notations (see Appendix A; also [25]; [18]): we write G for a generic group, G_{ab} if we know it is abelian, G_n if its order is n. For any group G, G' is the derived subgroup, $Z(G) \equiv Z_G$ is the center (centre), $G_{ab} = G/G'$ the abelianized group, $Inn(G) = G/Z(G)$ the group of inner automorphisms (*autos*), $Aut(G)$ the total group of *autos*, $Out(G) = Aut(G)/Inn(G)$ the group of classes (or types) of automorphisms (*outos*). $\#X$ at times would mark the number of elements in set X. $|.|$ means the order of some implicit group, $*...*$ would mean a higher statement, not shown why in the main text. We discuss often the automorphism group (for $G = G_{ab}$, abelian) and the group of classes of automorphisms, $Out(G)$, for G which, although computable from Ref. [29], is not calculated explicitly. For any n, $\phi(n)$ is the *Euler totient function*, which counts the numbers $\leq n$ coprime with it; for example, $\phi(9) = 6$. For the numeration of groups, we use nearly always the system in [29], which starts as $\underline{n/1}$ for the cyclic group Z_n for any number n, then the rest of the abelian groups of this order, then direct products, etc.

Some known results (proved or hinted at in the Appendices) we use directly are:

1. The dimension d_i of the *irreps* divides the order of the group, $|G| : d_i$.
2. The family of *irreps* of G includes that of all quotients $Q = G/K$.
3. The order of the abelianized also divides the order of the group (because $|G_{ab}| = N = |G|/|G'|$ and Lagrange's theorem).
4. If G abelian, $N = |G|$. If G is simple and non-abelian, $N = 1$.
5. If $|G| = p^f$, p a prime number, the center is non-trivial, $|Z_G| > 1$.
6. For $|Out(G)| > 1$, there must be repeated classes of same period in (1.2) for G.

From this and the number of classes r, we work out many of the characteristics of the group: after doing that, we continue with the usual characterization of groups by generators and relations, and consider several items: Hasse subgroup structure, Jordan-Hölder chain(s), Character Table, automorphisms ($Aut(G)$) and classes of them ($Out(G)$), etc., although we are not at all exhaustive; some natural extensions of groups, including the (semi)direct product(s), and the holomorph of a group, are also indicated. Possible outer *autos* relate different classes of the same type. When a group admits two or more representations with the same dimension, we may have (but not necessarily) isomorphism between them, related to *outos* of the group or of some quotient groups. In general $Aut(A)$ (A abelian) operates in the set (dual group) $\hat{A}$ of *irreps* of A, and $Out(G)$ operates in the set of *irreps* or the "dual" set $\hat{G}$. When actions of the group in some sets are important, we signal them also; $G \circ\!\!\rightarrow X$ will always mean for us: the group G operates in the set X.

2 Review of the paper

The plan of the work is as follows. In Sect. 3 we study groups with $|G| < 8$, showing simply why there are precisely 8 abelian +1 non-abelian = 9 groups, and its structure. Sect. 4 deals with groups of order 8 (five of them, 3 Abelian plus 2 non-). Sect. 5 studies other groups up to order 16, with order 12 left out, for Sect. 5A; in total so far there are 20 abelian +8 = 28 groups, $|G| < 16$. Sects. 6 and 7 deal with $|G| = 16$, with $5 + 9$ groups. Sects. 8 to 10 deal with the remaining groups, with $16 < |G| < 32$, again separating $|G| = 24$ (for Sect. 9). The total number of groups is 48 abelian plus 45 non abelian, a total of 93. Some Tables and final comments are grouped in Sects. 11 and 12.

We shall use also at times notions and notations of *category theory*: $\mathcal{G}$ stands for the category of all groups, and $\mathcal{G}^{\circ\circ}$ for that of finite groups; Ab for all abelian groups, Ab° for the finitely generated abelian groups (e. g. the integers Z), and $Ab^{\circ\circ}$ for that of the finite abelian groups. We include, of course, morphisms between the groups in each category. As said, we shall also use generally the notation of [29] for the number of the group: e.g. 4/2 is the *Vierergruppe* V, or second group of order 4; 24/1 is the group Z_{24}, 24/2 is $Z_{12} \times Z_2$ etc.

There are two Appendices; Appendix A deals with general definitions and results, around subsets (subgroups), extensions and natural maps (*morphisms*); then we deal with the structure of finite abelian groups, and next categorize the action of (finite) groups G on (arbitrary) sets Ω (orbits, fixed points, stabilizers, transitivity, etc); it also contains standard examples of families of groups (symmetric, cyclic, ..., dicyclic, Clifford, ...). Appendix B includes the necessary elements of Representation Theory: irreducibility, equivalence, Tables of Characters etc. Then we study all the possible groups up to two prime factors, $|G| = pq$; also we recall the finite fields $\mathbb{F}_q$ with $q = p^f$, power of a prime

p, and the $\mathbb{F}_q$-linear groups $GL_n(q)$ with some (sub)quotients (= quotients of subgroups). We deal finally with Coxeter groups [10], that is groups defined by reflections (involutions, $a^2 = e$).

The notation Sym_n or S_n, Alt_n, Z_n, D_n or Dih_n stands for the groups: Symmetric (order $n!$), Alternating ($n!/2$), Cyclic (n), and Dihedral ($2n$), $D_n = Z_n \rtimes Z_2$, with $\rtimes$ meaning semi-direct product; our D_n is labelled D_{2n} very often in the literature. Later we shall define and use the *dicyclic* groups Q_n, and also the *Clifford* groups Γ_n, Γ_n^+. As above, with $A \rtimes B$ we denote the semidirect product with kernel A (supposed abelian unless statement to the contrary); in particular $Hol(A) := A \rtimes Aut(A)$ is the *holomorph* (group) of abelian group A. For quotient groups $G/H = Q$, we shall use frequently the *exact sequence* formulation $H \to G \to Q$, explained in detail in App. A.

3 Groups up to order 8

The group of one element, e, will be noted I for 1/1, so $I = \{e\}$, and will be not too much used in the sequel, but it will count.

For order two we have only a group:
2/1: $Z_2 : \{a \mid a^2 = e\}$ abelian, cyclic and simple.

$$2 = 2 \cdot 1^2 = 1 \cdot 1_1(e) + 1 \cdot 1_2(a) \tag{3.1}$$

The two 1-dim *irreps* are $Id := D_0$ and D_0^-; in the second, $D_0^-(a) = -1$, of course, so it is *faithful*.

In any group G, elements $a \neq e$ with $a^2 = e$ are called involutions: each generates a different Z_2 (sub)group. We have the following four lemmas, all referred to the number 2, and very easy to prove:

Lemma 1.- $H \subset G$ and $[G : H] = 2 \Rightarrow H$ normal in G (only a coset, hence $gH = Hg$).

Lemma 2.- Z_2 normal in $G \Rightarrow$ central also (as $g \cdot Z_2 \cdot g^{-1} = Z_2 \Rightarrow g \cdot a \cdot g^{-1} = a$).

Lemma 3.- If, in some group G, all elements $\neq e$ are involutions, G is abelian and, if finite, $G \approx (Z_2)^n$. For ab involutive means $abab = e$, but $ab(ab)^{-1} = abba = e$ also, hence $ab = ba$; G has, if finite, order power of two, $G \approx (Z_2)^n$, so $|G| = 2^n$ with $2^n - 1$ involutions.

Lemma 4.- (Cauchy) $G \ni a$ involution, $\Rightarrow |G|$ even; and if $|G|$ even, # involutions odd number. *Proof*: If $\exists\, a$, $\exists\, Z_2$; now $Z_2 \subset G$, $\Rightarrow |G|$ even (Lagrange). Now if $|G|$ even, pair any g with $g^{-1} \neq g$: only involutions and e left over, so an even number; hence #

involutions is odd, and hence ≥ 1.

When G is a Z_2 *extension* of K, viz. when $K \to G \to Z_2$, we might write $G = K \cdot 2$; when G is a Q-*extension* of Z_2, viz. when $Z_2 \to G \to Q$, we can write $G = 2 \cdot Q$. This is the notation of the Atlas [12]; see also [16]. Some common examples follow:

1. $Sym_n/Alt_n = Z_2$, or $S_n = Alt_n \cdot 2$
2. $*O(n)/SO(n) = Z_2$, or $O(n) = SO(n) \cdot 2$ [$O(n)$, orthogonal; $SO(n)$, rotations]
3. $D_n/Z_n = Z_2$, or $D_n = Z_n \cdot 2$
4. $Z_2 \to SU(2) \to SO(3)$, or $SU(2) = 2 \cdot SO(3)$; in general
5. $Z_2 \to Spin(n) \to SO(n)$, *or* $Spin(n) = 2 \cdot SO(n)*$

As a fact, groups reflect symmetries of particular systems; for a general introduction to groups from this point of view, see [26]. See also the original [21].

The Coxeter diagram for Z_2 is simply $\circ$ (meaning a single generator a, with $a^2 = e$). The character Table and the Hasse subgroup structure are trivial here, so we shall show them later, in more complicated groups. As Z_2 is already simple, the Jordan - Hölder chain (J-H) is simply $\{Z_2, e\}$. There is only the Id automorphism in Z_2, as (e, a) must go to (e, a) under any *auto*: indeed Z_2 is the *only* group $G \in \mathcal{G}$ with no non-trivial *autos*:

Lemma 5.- $Aut(G) = I \Leftrightarrow G = Z_2 : Z_2$ is the ONLY group with not more than the trivial automorphism $\alpha = Id$.

Proof : if G is non-abelian, it has inner autos; and in any abelian group with *more* than involutions the map $\alpha : a \to a^{-1}$ is a non-trivial automorphism. Finally if e.g. is $A = (Z_2)^k$, $k > 1$ we have *autos*; in particular (see below, 4/2) if $A = Z_2^2 = V$, with involutions (a, b, ab), e.g. the map $\alpha : a \to b \to a$ is *auto. qed.*

Z_2 is also the additive group of the *finite field* $\mathbb{F}_2$; see App. B. Therefore, the *elementary abelian* group $(Z_2)^k$ can be written also as the *k-dim* vector space $\mathbb{F}_2^k$ over $\mathbb{F}_2$, with Automorphism group the linear group $GL_k(2)$; we have $GL_1(2) = I$, but we shall see that $GL_{2,3,4}(2)$ are important groups, which we shall encounter later; note $\mathbb{F}_2^* = \mathbb{F}_2 \backslash \{0\}$, the multiplicative group of the field, is just I.

Even order for *simple nonabelian* groups is the famous Feit-Thomson* result: any odd order group is *solvable*, hence cannot be simple. Also in the '60s, the search for centralizers of involutions (Brauer) were instrumental in completing the list of finite simple groups [14].

*The Feit-Thomson result (1963) [13]: any odd order group is solvable.

For order three 3, again we have only Z_3, as *abelian*, *cyclic* and *simple*.

$$3/1: \quad 3 = 3 \cdot 1^2 \,. - \quad 3 = 1 \cdot 1_1(e) + 2 \cdot 1_3(a, a^2). - \quad Z_3 : a; a^3 = e; Z_3 \equiv \{e, a, a^2\} \quad (3.2)$$

If $\mathbb{C} \ni \omega := \exp(2\pi i/3)$ (= rotation by 120°), besides the identical *irrep* $D_0 := \chi(a) = 1$ for Z_3, one faifthful *irrep* is $\chi(a) = \omega$, or $D_1 \equiv \chi(e, a, a^2) = (1, \omega, \omega^2)$; the other is the conjugate $D_2(...) = (1, \omega^2, \omega)$; this is related to $Aut(Z_3)$:

Now we have $Aut(Z_3) = Z_2$, namely $\alpha \neq Id$ is the map to the inverses for a, a^2 (resp. (a^2, a)). Hence the *holomorph* will have order 6; indeed, as we shall prove *later*,

$$Hol(Z_3) \equiv A \rtimes Aut(A)|_{A=Z_3} = Z_3 \rtimes Z_2 = S_3 = D_3. \quad (3.3)$$

We have also $Alt_3 = Z_3$, as $|Alt_3| = 6/2 = 3$. Z_3 is also the rotation symmetry of the regular triangle, $Rot(\triangle) = Z_3$. The involution α in $Aut(Z_3) = Z_2$, as hinted, also changes the two $D_{1,2}$ 1-dim *irreps* (out of the identical D_0): this is a general feature; as said, the *Aut* of an abelian group operates in its *irreps*. Finally $\mathbb{F}_3$ is the 3-element field; $GL_1(3) = \mathbb{F}_3^* = Z_2$ and, for example, $|PSL_2(3)| = (3^2 - 1)(3^2 - 3)/2/2 = 12$; in fact, $PSL_2(3) \approx Alt_4$, see later, and $|GL_2(3)| = 48$.

In any G, elements a, $a \neq e$, $a^3 = e$ will be called *cubic*; if G contains cubics, $|G| : 3$, again form Lagrange's. The number of cubics is even (if a cubic, $a^2 \neq a$ also), but the number of Z_3 subgroups might be even (including zero) or odd.

For order four $4 = 2^2$ we have only $4 = 4 \cdot 1^2$ from Burnside's, so group(s) are abelian; if $\exists a$, $a^4 = 1$, the group must be Z_4; if not, we have the group $V := Z_2 \times Z_2$ (Lemma 3), an *elementary abelian* group, as the only other possibility. Consider first the cyclic case Z_4:

$$4/1 \quad Z_4 = \{a \mid a^4 = 1\} = \{e, a, a^2, a^3\}. - \quad 4 = 1 \cdot 1_1(e) + 1 \cdot 1_2(a^2) + 2 \cdot 1_4(a, a^3) \quad (3.4)$$

The only nontrivial *auto* is the map $\alpha : a \to a^{-1} = a^3$, so $Aut(Z_4) = Z_2$; hence $Hol(Z_4) := Z_4 \rtimes Z_2 = D_4$, a *dihedral group*, see below. The (unique) Jordan-Hölder (JH) chain is $\{Z_4, Z_2, e\}$.

Z_4 has four *irreps* 1-dim, of course. The two *faithful* ones are $\chi(e, a, a^2, a^3) \Rightarrow (1, \pm i, -1, \mp i)$; besides the identical *irrep*, the map $Z_4 \to Z_2$ provides the fourth one, unfaithful. The Hasse subgroup diagram is simply:

$$\begin{array}{c} \text{(all) } Z_4 \text{ (order 4)} \\ | \\ (a^2)\ Z_2 \text{ (order 2)} \\ | \\ (e)\ I \end{array} \quad (3.5)$$

Z_4 is the rotation symmetry group of the square $\Box$, and the isometry group of the *Swastika*.

Consider now the *Vierergruppe* V of F. Klein (or four-group) $\approx Z_2^2$:

$$\underline{4/2} \text{ or } V : \{e, a, b, ab\} \text{ with } a^2 = b^2 = (ab)^2 = e. - \quad 4 = 1 \cdot 1_1(e) + 3 \cdot 1_2(a, b, ab) \quad (3.6)$$

It has NO faithful *irreps*; besides the identical D_0, the other three are $(1, a', b', a'b')$ with $a' = \pm 1$, $b' = \pm 1$. The reason is, there are three quotients $V \to Z_2$, which have to be represented by their (faithful) *irrep*.

The Hasse diagram is ($3^* Z_2$: there are three subgroups type Z_2)

$$\begin{array}{c} V \\ | \\ 3^* Z_2 \\ | \\ I \end{array} \quad (3.7)$$

As Automorphism group, any $\alpha \in Aut(V)$ must permute the involutions: ($\alpha_1(a) = b$, $\alpha_1(b) = a$) generates Z_2, but ($\alpha_2(a) = b$ and $\alpha_2(b) = ab$) generates Z_3: hence $Aut(V) = S_3$, which creates the holomorph of V as a rather large (Hol) group:

$$Hol(V) := V \rtimes S_3 = S_4 \text{ (as we shall see), order } 4 \times 6 = 24 \quad (3.8)$$

Now V can be considered also as $\mathbb{F}_2^2$, the 2-dim vector space over the finite field $\mathbb{F}_2$ (App. 2). For $V \approx \mathbb{F}_2^2$, the automorphism group must be $GL_2(2)$: we shall exploit later the natural isomorphism (as we know $GL_2(2)$ cannot be abelian)

$$GL_2(2) \approx S_3 = D_3 \quad \text{as order} \quad (2^2 - 1)(2^2 - 2) = 3 \cdot 2 = 3 \cdot 2 \cdot 1 = 6. \quad (3.9)$$

Finally, the Coxeter *diagram* for V is just two unconnected balls: o o (so $G = Z_2 \times Z_2$). As subgroup of Alt_4, $V \subset Alt_4$ contains permutation (12)(34), etc.). Finally, recall the three extensions $V \rtimes Z_2$ ($\approx D_4$, see below Sect. 4), $V \rtimes Z_3$ ($\approx Alt_4$, no elements of order 4), $V \rtimes S_3$ ($\approx Sym_4$, see below Sect. 9).

Order five, 5 prime number, so only a group:

$\underline{5/1}$ or $Z_5 : \{a \mid a^5 = e\}$.- Finite, abelian, cyclic and simple. $5 = 5 \cdot 1^2 = 1 \cdot 1_1 + 4 \cdot 1_5$

$Aut(Z_5) = Z_4$, as any *auto* α maps a to any power a^k, $k \neq 0$. So we have $Hol(Z_5) := Z_5 \rtimes Z_4$, order 20.- In particular it contains $D_5 := Z_5 \rtimes Z_2$, order 10.

If $\phi = exp(2\pi i/5)$ = rotation by 72°, and D_0 is the identical *irrep*, the second D_1 can be defined by $a \to \phi$, and the other three $\neq id$ as variations, e.g. $a \to \phi^2$ etc., permuted cyclically by $Aut(Z_5) = Z_4$, see below in detail for Z_7.

Z_5 is the rotation symmetry group of the regular pentagon ⬠. Regular polytopes (generalization of *regular* polyhedra in any dimension) might have pentagonal faces in 2, 3 and 4 dimensions *only*.

Groups of order six. There is a single abelian one, Z_6, as $6 = 2 \cdot 3$, primes, and one has the cyclic group

$$6/1 \quad Z_6 : \{a \mid a^6 = e\}. - \quad Z_6 = Z_2 \times Z_3, \quad \text{or} \quad \{b, c \mid b^3 = c^2 = e; bc = cb\} \tag{3.10}$$

The equations type ($Z_6 = Z_2 \times Z_3$) are NOT true in general (e.g. $Z_4 \neq \{Z_2 \times Z_2\}$; it is true for $Z_p \times Z_q$, with $p \neq q$ primes). The isomorphism in (3.10) $\iota : Z_6 \approx Z_2 \times Z_3$ is proved e.g. by $\iota(a) = bc$.

$|G| = 6$ group(s), non-abelian. One writes $6 = N \cdot 1^2 + M \cdot 2^2$, as $3^2 = 9 > 6$. Now $6 > N \geq 1$ if G nonabelian, and N divisor of 6: the only solution is $6 = 2 \cdot 1^2 + 1 \cdot 2^2$: so # classes $\equiv r = 3$, there are possibly just *three* classes, and the abelianized group being of order two, it has to be Z_2, and then the derived G' has to be Z_3. G of order 6 must have elements $\neq e$ of order 2 and 3, the number of 3 even ($a \neq a^2$), but that of 2 odd (Cauchy; see our Lemma 3 in 2/1); hence, as Z_3 normal, the only solution is 2 elements of order 3 and 3 of order 2. That is, in classes

$$6 = 1 \cdot 1_1 + 1 \cdot 3_2 + 1 \cdot 2_3 \tag{3.11}$$

Hence Z_3 is a unique subgroup and so must be normal (Sylow), and it is the derived group G': so from $G' \to G \to G_{ab} = Z_2$, it is $Z_3 \to G \to Z_2$; now $Aut(Z_3) = Z_2$, hence there is a unique solution $D_3 := Z_3 \rtimes Z_2$, order $3 \cdot 2 = 6$; so there is only a group, which of course is the smallest nonabelian group:

$$6/2 \quad D_3 = Z_3 \rtimes Z_2 = S_3; \quad \{a, \alpha \mid a^3 = \alpha^2 = 1, \alpha a \alpha^{-1} = a^2 = a^{-1}\} \tag{3.12}$$

Now, trivially, $3 \cdot 2 = 3 \cdot 2 \cdot 1$, and one shows at once that $D_3 = S_3$: the 3 order-two elements are the permutations (12), (23) and (13), and the 2 order-3 are (123) and (132).

The full group diagram is simply:

$$\begin{array}{ccccc} & & I & & \\ & & \downarrow & & \\ Z_3 & \longrightarrow & S_3 & \longrightarrow & Z_2 \\ & & \downarrow & & \\ & & S_3 & \longrightarrow & S_3 \end{array} \qquad \text{(i.e., no } outos\text{)} \tag{3.13}$$

To see another form, take again $\mathbb{F}_2$, the Galois field of order two $\{1, 0\}$. The projective line (App. B) $\mathbb{F}_2P^1$ has three elements (say $0, 1, \infty$), and indeed the projective group $PGL_2(2) = GL_2(2)$ is S_3 also, because $\mathbb{F}_2^2 \approx V$ (see 4/2). Notice, for the field $\mathbb{F}_2$, $GL_n(2) = PGL_n(2) = SL_n(2) = PSL_n(2)$ (because $\mathbb{F}_2^* = \{1\}$).

Finally, as isometry of regular triangles $\triangle$, Z_3 is the rotation group and S_3 the full isometry group... The Coxeter diagram for S_3 is clearly o – o, as any link with no mark is supposed to be of period 3 (Cfr. App. B): the group is also formed with $\{a, b\}$ as reflections, by $a^2 = b^2 = (ab)^3 = e$.

So for the smallest nonabelian group we have many characterizations, to repeat:

$$D_3 = S_3 = Z_3 \rtimes Z_2 = Sym_3 = PSL_2(2) = GL_2(2) = Hol(Z_3) = Z_3 \rtimes Aut(Z_3) \tag{3.14}$$

Regular hexagons $\hexagon$ tessellate the plane optimally; this is probably why honeycombs by bees are made out of hexagons...

The definition for Z_7 is of course $\{a \mid a^7 = e\}$, with

$$7/1 \quad Z_7 : \{7 = 7 \cdot 1^2\}. - \quad 7 = 1 \cdot 1_1 + 6 \cdot 1_7 \quad \text{again abelian, cyclic and simple} \tag{3.15}$$

What about automorphisms α? They are defined by the image of generator $a : \alpha(a)$; as the map might be any power $\alpha(a) = a^k$, $k \neq 0$, the group is $Aut(Z_7) = Z_{p-1|p=7} = Z_6 = Z_2 \times Z_3$. Hence the Holomorph $A \rtimes Aut(A)$ is $Z_7 \rtimes Z_6$, of order $7 \cdot 6 = 42$. Of course, the Dihedral extension is included, $D_7 \subset Hol(Z_7)$, as $D_7 = Z_7 \rtimes Z_2$, where the single non-trivial *auto* is only $\alpha(a) = a^{-1} = a^7$, as we shall see; compare discussion after 5/1.

We include the whole Character Table for completeness; if $\Phi \equiv$ rotation by $2\pi/7 \approx 51°$, it is

Z_7 is the rotation symmetry of the regular heptagon, and D_7 the full isometry group (rotations and reflections) of the same; it is also the periodicity of the white keys in the *piano* musical instrument.

7 is also a number with religious resonances: the seven arms of the jewish candelabrum; or the seven days of creation (Jews and Christians), the 7-day week, etc.

	1A	7A	7B	7C	7D	7E	7F
χ_0	1	1	1	1	1	1	1
χ_1	1	Φ	Φ^2	Φ^3	Φ^4	Φ^5	Φ^6
χ_2	1	Φ^2	Φ^4	Φ^6	Φ	Φ^3	Φ^5
χ_3	1	Φ^3	Φ^6	Φ^2	Φ^5	Φ	Φ^4
χ_4	1	Φ^4	Φ	Φ^5	Φ^2	Φ^6	Φ^3
χ_5	1	Φ^5	Φ^3	Φ	Φ^6	Φ^4	Φ^2
χ_6	1	Φ^6	Φ^5	Φ^4	Φ^3	Φ^2	Φ

Table 1.— Character Table

4 Order 8

Let us work out now possible groups of order 8. First, there are the three abelian cases, as $8 = 2^3$ and Part(3) = 3: [3], [2, 1] and [1, 1, 1], see App. A. If there is an element of order 8, it generates Z_8, the *cyclic* group; if all elements $\neq e$ are involutions, the group is $(Z_2)^3$, an *elementary abelian* group; and if there are 4-th order elements (but not 8-th), the abelian group has to be $Z_4 \times Z_2$:

$$G_8, \qquad \text{Abelian}: \quad Z_8; \quad Z_4 \times Z_2; \quad (Z_2)^3 \tag{4.1}$$

$$\underline{8/1} \quad Z_8: \quad \{a \mid a^8 = e\}. - \quad 8 = 1 \cdot 1_1(e) + 1 \cdot 1_2(a^4) + 2 \cdot 1_4(a^2, a^6) + 4 \cdot 1_8(\text{rest}) \tag{4.2}$$

Let us look for its *automorphism* group; as Z_8 is cyclic, any auto α is defined once $\alpha(a)$ is known; $\alpha(a)$ can be a, a^3, a^5 or a^7: these $\alpha \neq Id$ are *involutions* (and commute), e.g. if $\alpha(a) = a^5$, $\alpha(a^5) = a^{25} \equiv a \Rightarrow \alpha^2 = Id$. Therefore

$$Aut(Z_8) = (Z_2)^2 = V \tag{4.3}$$

Because this, we shall see that Z_8 admits *three* different Z_2 extensions, not only the conventional Dihedral $D_8 = Z_8 \rtimes Z_2$ (see order 16, Sect. 7).

As for $A = (Z_2)^3$, it is the 3-dim vector space over the $\mathbb{F}_2$ field,

$$\underline{8/3} \quad (Z_2)^3 \approx \mathbb{F}_2^3 = (a, b, c). - \quad 8 = 1 \cdot 1_1(e) + 7 \cdot 1_2(a, b, c; ab, ac, bc; abc). \tag{4.4}$$

Hence the Automorphism group is some matrix group:

$$Aut((Z_2)^3) = Aut(\mathbb{F}_2^3) = GL_3(2) \equiv PSL_3(2) = PSL_2(7) \tag{4.5}$$
$$\text{order } (2^3-1)(2^3-2)(2^3-4) = 168 = (7^2-1)(7^2-7)/(7-1)/2$$

168 is the order of the *second smaller* nonabelian simple group (the *first* is Alt_5); for a modern outlook see [15]. Both $GL_3(2)$ and $PSL_2(7)$ are of same order and simple (see Appendix B), to they must be *isomorphic* (the smallest number for two different *simple* group of *same* order is bigger than 20000: see (Artin [1])). If $\{a, b, c\}$ generate $(Z_2)^3$, in any auto $\alpha : a \to \alpha(a)$, a can go to 7 positions, then b only 6, as $\alpha(b) \neq \alpha(a)$, but c only four, as a, b and ab have already prefixed positions; hence directly we obtain $|Aut(Z_2^3)| = 7 \cdot 6 \cdot 4 = 168$. Now the group $PGL_2(7)$ acts naturally and sharp 3-transitive (App. B) in the $8(= 7 + 1)$-point projective line $\mathbb{F}_7 P^1$, hence is of order $(49 - 1)(49 - 7)/6 = 336$, so $|PSL_2(9)| = 168 = 336/2$. Recall $Aut(Z_2^2) = S_3$, arbitrary permutations of a, b and ab. But here, $|Aut(Z_2^3)| = 168 \ll |S_7| = 5040$; *for analogous reasons, if $\mathbb{H}$ and $\mathbb{O}$ are the division algebras of the *quaternions* and *octonions*, resp., $Aut(\mathbb{H}) = SO(3)$, while $Aut(\mathbb{O}) = G_2 < (!)SO(7)$: as Lie groups, $dim\, G_2 = 14 < dim\, SO(7) = 21^*$.

Next we have $Z_4 \times Z_2$: suppose is generated by a and b ($a^4 = b^2 = e$; $ab = ba$); then

$$\underline{8/2} \quad Z_4 \times Z_2. - \quad 8 = 1 \cdot 1_1(e) + 3 \cdot 1_2(a^2, b, ba^2) + 4 \cdot 1_4(a, a^3, ba, ba^3) \tag{4.6}$$

As for $Aut(Z_4 \times Z_2)$, as any *aut* element α keeps order, so it must map a to (a, a^3, ba or ba^3), while b maps to (b, or ba^2): the possible map b to a^2 is fixed for $\alpha(a)$ (the inverse): so we have $4 \cdot (3-1)$ cases: *eight* possible *autos*; they go from $\alpha_0 = id$, $\alpha_1(a, b) = (a, a^2 b)$; $\alpha_2(a, b) = (a^3, b)$ ... up to $\alpha_7(a, b) = (a^3 b, a^2 b)$. α_5 and α_7 have order four, the rest two. Hence the Aut group has to be $D_4 := Z_4 \rtimes Z_2$, (see Sect. 7) as the only rival, $Z_4 \times Z_2$ itself, has four elements of order 4, see (4-6). This remarkable result is also proved directly in [32]:

$$Aut(Z_4 \times Z_2) = D_4 \tag{4.7}$$

The Hasse diagram for $(Z_4 \times Z_2)$ is worth to describe ([29], type $\underline{8/2}$):

$$\begin{array}{ccccc} & & Z_4 \times Z_2 & & \\ & \swarrow & & \searrow & \\ 2^* Z_4 & & & & 3^* Z_2 \\ & \searrow & & \swarrow & \\ & & I(e) & & \end{array} \tag{4.8}$$

Classes for order 8: call $n_{1,2,4,8}$ the number of elements of each order in any G, $|G| = 8$; we have $n_1 = 1$ in all cases; in the nonabelian ones, $n_8 = 0$ (otherwise the group is Z_8), and $n_2 < 7$ (otherwise the group is $(Z_2)^3$). On the other hand, n_2 is odd, and n_4 even (because Lemma 4 above and: if a is order four, $a^3 \neq a$ is also).

Non abelian groups of order 8; Burnside's relation is *unique*, as in $8 = N \cdot 1^2 + M \cdot 2^2$ is $N < 8$ if nonabelian: so

$$8 = 4 \cdot 1^2 + 1 \cdot 2^2 \qquad (r = \#\text{classes} = 5) \tag{4.9}$$

Hence, the derived group G' is $= Z_2$, implied by abelianized G_{ab} of order 4; but now, derived Z_2 implies Center contains Z_2, because Z_2 normal implies Z_2 central (see Lemma 2 in Sect. 3); and we have to solve for number of involutions (n_2) and number of elements of order four (n_4). n_2 has to be odd (Cauchy, see Lemma 3.); the fourth order elements go in pairs (b, b^3); so we have *three* cases:

$$(n_2, n_4) = (1, 3 \cdot 2), \quad \text{or} \quad (3, 2 \cdot 2) \quad \text{or} \quad (5, 1 \cdot 2) \tag{4.10}$$

In the *third* case, the Z_4 is unique, so normal (Sylow), and the solution of $Z_4 \to G_8 \to Z_2$ is clearly $G_8 \equiv D_4 = Z_4 \rtimes Z_2$: the Dihedral group (of order 8), or 8/4. In the *first case*, if i and j are order-four, the squares coincide, and ij must be also of order four: the group is called the Quaternion group 8/5, specified as $Q := \pm(1, i, j, ij)$, with $i \cdot j \cdot i^{-1} = -j$ etc. One writes a quaternion (Hamilton, 1842) as $q = u + xi + yj + z(ij) \in \mathbb{H},\ x, y, z \in \mathbb{R}$.

As for the second solution in 4.10, it leads to $Z_4 \times Z_2$ or 8/2 of above, and to nothing else. Hence the two nonabelian groups of order 8 are

$$D_4 = Z_4 \rtimes Z_2 : (n_2, n_4) = (5, 2) \qquad \text{and} \qquad Q : (n_2, n_4) = (1, 6) \tag{4.11}$$

In both nonabelian cases the center is Z_2: this is to be expected, as in any p-Group ($=$ order p^f) the center is never trivial [25]. D_4 is the full symmetry group of the regular square; it is also called *octic group*. Q is the quaternion group, also called Q_2 (dicyclic) and Γ_2 (Clifford) (see just below).

For the octic group D_4, the class split is, as $D_4 = Z_4(a) \rtimes Z_2(\alpha)$:

$$8/4 \qquad 8 = 1 \cdot 1_1(e) + 1 \cdot 1_2(a^2) + \bar{2} \cdot 2_2(\alpha, \alpha a^2; \alpha a, \alpha a^3) + 1 \cdot 2_4(a, a^3), \tag{4.12}$$

hence it admits a *unique* involutive *outo* (changing classes in $2 \cdot 2_2$): $Out(D_4) = Z_2$; however, it *cannot* be implemented as semidirect product, as one shows that the inverse image of that Z_2 in $Aut(D_4)$ is Z_4, not Z_2. We have $|Aut(D_4)| = |Inn| \cdot |Out| = 8$; indeed, one shows $Aut(D_4) = D_4$, inspite having a center! (Compare $S_3 = Aut(S_3)$ above; this peculiarity of D_4 is noted also by Robinson [25] p. 30). In other words, one has the curious result

$$Z_4 \rtimes Z_2 = D_4 = V \rtimes Z_2 \tag{4.13}$$

which can be proved directly as isomorphism $\iota : (e, b, a^2, ba^2)$ makes up D_4 as $V \rtimes Z_2$.

As for $Q = \pm(1, i, j, ij = k)$, the class split is different,

$$8/5 \qquad 8 = 1 \cdot 1_1(e) + 1 \cdot 1_2(-1) + \bar{3} \cdot 2_4(\pm i, \pm j, \pm ij) \tag{4.14}$$

($-1 = i^2$ etc.) and hence the Aut group can be up to S_3; it is S_3, as one can map i to three values i, j, ij, and then j to two only, so $|Aut| = 3 \cdot 2$, and being non-abelian, it has to be S_3.

For another construction of Q from Z_4, see [32]

In App. A (examples) we gave the definitions of *Clifford* Γ_n [6] and *dicyclic* groups Q_n; that accepted, we can write at once

$$Q = \Gamma_2, \text{ and } \Gamma_2^+ = Z_4 \qquad .- \quad Q = Z_4 \rtimes_{/2} Z_4 \equiv Q_2 \tag{4.15}$$

Recall (App. A and B) a (finite) p-Group is a group of order p^f. A p-group is called *extraspecial*, if the center $Z(G)$ is Z_p and the quotient $Inn(G) := G/Z(G)$ is the elementary abelian group $(Z_p)^{f-1}$; one shows [14] that f is odd, $f = 1+2n$. In our simple case $8 = |G| = 2^{1+2\cdot 1}$ we see our two groups D_4 and Q are extraspecial (center Z_2 and quotient V in both cases); notice also G' and G/G' behave similarly as $Z(G)$ and $G/Z(G)$. Extraspecial groups are important because they show up as *centralizers of involutions*, and as such they were instrumental for the big advances in the '60 and '70s in the search for finite simple groups [14], [24].

This completes the study of the $(3+2)$ 8-order groups. Notice how fast the number of groups grows with the prime factors in the order: if $|G| = p$, only 1 group; if $|G| = pq$, at most two; and if $|G| = pqr$, at most five (as we shall see); generally the number of groups with four factors is 15. So up to order ≤ 8, we have in total 14 groups, 3 non-abelian.

5 Groups with order $8 < |G| < 16$

Exclude $|G| = 12$ for the moment (studied at the end, Sect. 5A). From now on we shall use directly some results of the Appendices; so for example, for orders $|G| = p$, p^2 or pq, we shall take directly the results for granted.

Order 9.- $|G| = 9$; as $3^2 = 9 \cdot 1^2$ only possibility, the groups are abelian, and there are *two* of them, as $9 = 3^2$ and $Part(2) = 2$, namely Z_9 and $(Z_3)^2$.

$$9/1 \quad Z_9.-\{a \mid a^9 = e\} \qquad 9 = 1 \cdot 1_1(e) + 2 \cdot 1_3 \ (a^3 \text{ and } a^6) + 6 \cdot 1_9 \ (\text{rest}) \tag{5.1}$$

so the group is (abelian and) cyclic. As for *autos*, $|Aut(Z_9)| = \phi(9) = 6$: the function of Euler $\phi(n)$, counting the number of coprimes with n (here 2, 4, 5, 7, 8 and 9). One sees easily that $\alpha(a) = a^2$ is of order 6, hence

$$Aut(Z_9) = Z_6 \qquad \alpha(a) = a^2 \tag{5.2}$$

As for the other group, $Z_3 \times Z_3$, it is an elementary abelian group, $\approx \mathbb{F}_3^2$, with $\mathbb{F}_3$ the (Galois) field with 3 elements, and therefore

$$9/2 \qquad Z_3 \times Z_3 \approx \mathbb{F}_3^2 \qquad .- \qquad 9 = 1 \cdot 1_1(e) + 8 \cdot 1_3 \tag{5.3}$$

$|Aut(Z_3 \times Z_3)| = |GL_2(3)| = (3^2-1)(3^2-3) = 48$. $Hol(Z_3^2) = Z_3^2 \rtimes G_{48} = G_{432}$. As reasoning, if Z_3^2 has points (elements) $(e; a, a^2; b, b^2; ab, a^2b, ab^2, a^2b^2)$, under *auto* α a can go to eight points, but then b to six: $\alpha(a)$ and $\alpha(a^2)$ excluded; hence, order $8 \cdot 6 = 48$.

Order ten.- As $10 = 2 \cdot 5$, both primes, there are only the two known solutions, namely the cyclic $Z_{10} = Z_5 \times Z_2$ and the Dihedral, $D_5 = Z_5 \rtimes Z_2$:

$$\underline{10/1} \quad Z_{10} : \{a \mid a^{10} = e\};\ Z_{10} = Z_5 \times Z_2. - \ 10 = 1 \cdot 1_1(e) + 1 \cdot 1_2(a^5) + 4 \cdot 1_5(a^{2,4,6,8}) + 4 \cdot 1_{10} \tag{5.4}$$

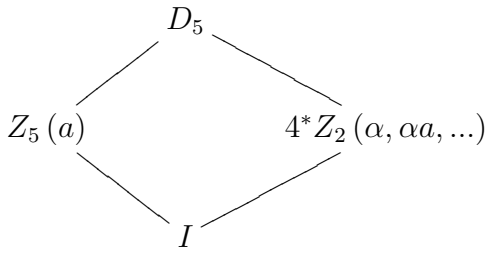
The $Aut(Z_{10})$ group is easy to compute: as 2, 5 are prime,

$$Aut(Z_{10}) = Aut(Z_5 \times Z_2) = Aut(Z_5) \times Aut(Z_2) = Z_4 \times \{e\} = Z_4 \tag{5.5}$$

Hence e.g. $|Hol(Z_{10})| = 40$.

$$\underline{10/2} \qquad \begin{array}{l} D_5 = Z_5 \rtimes Z_2. - \ (a^5 = \alpha^2 = 1; \alpha \cdot a \cdot \alpha^{-1} = a^{-1}, \text{ or } \alpha \cdot a \cdot \alpha = a^4). \\ 10 = 2 \cdot 1^2 + 2 \cdot 2^2. - \quad 10 = 1 \cdot 1_1(e) + 1 \cdot 5_2 + \bar{2} \cdot 2_5 \end{array} \tag{5.6}$$

There is clearly a leftover *auto*, as $|Hol(Z_5)| = 20$: $Out(D_5) = Z_2$, changing the two order-5 classes. The complete structure is given by the diagram



(5.7)

11/1 Order 11 .- As 11 is prime, only $\exists\ Z_{11}\ (a \mid a^{11} = e)$, with $Aut(Z_{11}) = Z_{10} = Z_2 \times Z_5$. If α is the generating *auto*, $\alpha(a) = a^2$, etc. As before, Z_{11} is abelian, cyclic and simple.

13/1. Same with 13, another prime, Z_{13}; and, of course, $Aut(Z_{13}) = Z_{12} = Z_4 \times Z_3$.

Order 14.- The abelian case, Z_{14} is of course $\approx Z_7 \times Z_2$ and unique; there is also the dihedral D_7, as the only non-abelian; so *in toto*:

$$\underline{14/1} \quad Z_{14} = Z_7 \times Z_2. - \quad \text{Abelian and cyclic.} - \quad Aut(Z_{14}) = Aut(Z_7) = Z_6 \tag{5.8}$$

$\underline{14/2}$ $\quad D_7 = Z_7(a) \rtimes Z_2(\alpha),\ a^7 = \alpha^2 = 1,\ \alpha \cdot a \cdot \alpha = a^6. -\ 14 = 2 \cdot 1^2 + 3 \cdot 2^2. -$

$$14 = 1 \cdot 1_1(e) + 1 \cdot 7_2(\alpha, \alpha a, \cdots, \alpha a^6) + \bar{3} \cdot 2_7 \tag{5.9}$$

What about $Out(D_7)$? It must be Z_3, as the three order-7 classes can be mixed, but in order. Hence

$$|Aut(D_7)| = |Inn| \times |Out| = 14 \times 3 = 42 \tag{5.10}$$

as D_{14} has trivial center, but derived group is Z_7 (non-trivial, as in any Dihedral). The last equation corresponds of course to $|Hol(Z_7)| = 42$, $Hol(Z_7) = Z_7 \rtimes Z_6$.

Order 15.- As $15 = n \cdot 1^2 + m \cdot 3^2$, with $n = 1, 3, 5$ or 15, only solution is $n = 15$, $m = 0$, which gives

$\underline{15/1}$ $Z_{15} = Z_5 \times Z_3$ as the only possibility. We say that primes (3, 5) are *incompatible* (by contrast, we shall see that (3, 7) are compatible). As (5, 3) =1, the *Aut* group just factorizes:

$$Aut(Z_{15}) = Aut(Z_3) \times Aut(Z_5) = Z_2 \times Z_4 \qquad |Hol(Z_{15})| = 15 \cdot 8 = 120 \tag{5.11}$$

Sect 5·A .- Groups of Order 12.- $12 = n \cdot 1^2 + m \cdot 2^2 + r \cdot 3^2$ has *three* solutions for the first time for the Burnside relation:

$$12 = 12 \cdot 1^2\ (Z_{12}); \quad 12 = 4 \cdot 1^2 + 2 \cdot 2^2\ (D_6); \quad \text{and} \quad 12 = 3 \cdot 1^2 + 1 \cdot 3^2\ (Alt_4); \tag{5.12}$$

we wrote an example of each, to be discussed next. As $12 = 2^2 \cdot 3$, there are two abelian groups: $Z_{12} = Z_4 \times Z_3$ (our example) and $V \times Z_3$: so $\underline{12/1}$ and $\underline{12/2}$ are cleared up:

$$\underline{12/1} = Z_{12} \qquad Aut(Z_{12}) = Aut(Z_4) \times Aut(Z_3) = Z_2 \times Z_2 = V \tag{5.13}$$

$$Z_{12} = Z_4 \times Z_3 \qquad 1 \cdot 1(e) + 1 \cdot 1_2(a^2) + \bar{2} \cdot 1_3(b, b^2) + \bar{2} \cdot 1_4(a, a^3) + 2 \cdot 1_6(a^2 b, a^2 b^2) + 4 \cdot 1_{12}(ab, ab^2, a^3 b, a^3 b^2) \tag{5.14}$$

$$\underline{12/2} = V \times Z_3 = Z_2 \times Z_6 \qquad Aut(V \times Z_3) = S_3 \times Z_2 \text{ (order 12, so it is } \underline{12/4} \text{ below)} \tag{5.15}$$

$$1 \cdot 1_1(e) + 3 \cdot 1_2(a, b, ab) + 2 \cdot 1_3(c, c^2) + 6 \cdot 1_6(a, ab^2, b, bc^2, abc, abc^2) \tag{5.16}$$

As for the non-abelian groups, see first that $12 = 3 \cdot 1^2 + 1 \cdot 3^2$, with four classes, implies $G_{ab} = 3$, so $G_{ab} = Z_3$, and hence $G' = Z_4$ or V: only V to form the non-abelian case, as $Aut(Z_4) = Z_2$, $\neq Z_3$ but $Aut(V) = S_3 \supset Z_3$; so we have

$$\underline{12/3} \qquad G_{12} : \; V \to G_{12} \to Z_3 \text{ or } G_{12} := V \rtimes Z_3 = Alt_4, \\ 12 = 1 \cdot 1_1 + 1 \cdot 3_2 + \bar{2} \cdot 4_3 \tag{5.17}$$

as Alt_4 has the class of $(12)(34)$, that closes to V, and the class of (123), which closes to Z_3. The repetition $\bar{2}$ hints at the extension to the symmetric group S_4, of course, which is $S_4 = Alt_4 \cdot 2$ in the Atlas [12] notation.

As for $12 = 4 \cdot 1^2 + 2 \cdot 2^2$ (six classes), now Z_3 is the derived, and V or Z_4 the abelianized; *both* give solutions, as

$$Z_3 \rtimes Z_4 = Q_3 \quad \text{and} \quad Z_3 \rtimes V = S_3 \times Z_2 = Z_6 \rtimes Z_2 = D_6 \tag{5.18}$$

As said, this is the first case in which there are more than one (two) possibilities for non-abelian Burnside's. In the first, $G_{ab} = Z_4$, so the construction must be

$$Z_3 \to G_{12} = Q_3 \to Z_4$$

As for $12 = 4 \cdot 1^2 + 2 \cdot 2^2$, it is also trivially realized in $Z_2 \times S_3 = (b, a, \alpha)$. The class splits are, first

$$\underline{12/4} \quad Z_2 \times S_3 . - \; 12 = 1 \cdot 1_1 + 1 \cdot 1_2(b) + 2 \cdot 3_2(\alpha, \alpha a, \alpha b) + 1 \cdot 2_3(a, a^2) + 1 \cdot 2_6(ba, ba^2) \tag{5.19}$$

Notice $\underline{12/3} = Alt_4$ has no elements of order 6, while $\underline{12/4}$ none of order 4: the reciprocal of Lagrange theorem is *not* true. $Out(Alt_4) = Z_2$ necessarily.

And secondly, there is the dicyclic Q_3 group, or

$$\underline{12/5} \; Q_3 = Z_6 \rtimes_{/2} Z_4 = Z_3 \rtimes Z_4 : \quad 12 = 1 \cdot 1_1 + 1 \cdot 1_2 + 1 \cdot 2_3 + \bar{2} \cdot 3_4 + 1 \cdot 2_6 \tag{5.20}$$

We see there is again only a nontrivial *outo*, mixing the two order-3 classes.

This completes our study of groups up to order 16. As final result, for order less than 16, we find 20 abelian groups plus 8 non-abelian. The non-abelian ones, with their *outos*, are given in Table 2:

Order	Group G	$\lvert Out\, G\rvert$
6	S_3	1
8	D_4	2
8	Q	6
10	D_5	2
12	Alt_4	2
12	$D_6 = S_3 \times Z_2$	2
12	$Q_3 = Z_3 \rtimes Z_4$	2
14	$D_7 = Z_7 \rtimes Z_2$	3

Table 2.— $|Out(G)|$ for $|G| < 16$

6 Abelian Groups of order 16

From order 16 on we shall be less exhaustive: we shall take the groups for granted, explaining them with generators and relations. Cfr. e.g. [10] or [29]; see also the modern treatment [32]. The *Aut* groups will also be indicated only succinctly.

As $16 = 2^4$ and $Part(4) = 5$, ($[4], [3,1], [2^2], [2,1^2]$ and $[1^4]$), there are FIVE abelian groups of order 16. They are (Appendix A):

$$Z_{16}, \quad Z_8 \times Z_2, \quad Z_4 \times Z_4, \quad Z_4 \times Z_2^2 = Z_4 \times V \quad \text{and} \quad Z_2^4. \tag{6.1}$$

The first one is the only cyclic, and the last is *elementary abelian*. The third is like $(Z_4)^2$, understood as modulus over Z_4 as a *ring* (not as Galois field!). Now in detail:

16/1 $Z_{16} = \{a \mid a^{16} = e\},\ 16 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 1_4 + 4 \cdot 1_8 + 8 \cdot 1_{16}$: abelian cyclic.

The *Aut* group has order 8, as $|Aut(Z_n)| = \phi(n)$, Euler's function; $\phi(16) = 8$. It has to be $Z_4 \times Z_2$, as it has four elements of order 4, 3 of order 2, and none of order 8. See this group above.

16/2 $Z_8 \times Z_2 = \{a, b \mid a^8 = b^2 = e; ab = ba\},\ 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 4 \cdot 1_4 + 8 \cdot 1_8.$

The *Aut* group has order $16 = 8 \cdot 2$, as a can go 8 places, and then b to $3 - 1$; it turns out that $Aut(Z_4 \times Z_2) = D_4 \times Z_2$ o 16/6 (see later).

16/3 $Z_4 \times Z_4 \approx (Z_4)^2,\ 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 12 \cdot 1_4;\ |Aut\,((Z_4)^2)\,| = 96$, as in (a, b) of fourth order generating 16/3, a can go to 12 positions, but then b only to eight. One shows also $Aut\,((Z_4)^2) = GL_2(Z_4)$ as 2×2 matrices over the *ring* Z_4.

16/4 $Z_4 \times {Z_2}^2 = Z_4 \times V;\ 16 = 1 \cdot 1_1 + 7 \cdot 1_2 + 8 \cdot 1_4;\ |Aut(Z_4 \times {Z_2}^2)| = 192$, as, if a, b, c generate Z_4, Z_2 and Z_2, a has 8 choices, for 6 of b and 4 of c.

16/5 $Z_2{}^4 : 16 = 1 \cdot 1_1 + 15 \cdot 1_2$.

As before $Z_2{}^3$ for order 8, this *elementary abelian group* is a 4-dim vector space over the *field* $\mathbb{F}_2$. So $Z_2{}^4 \approx \mathbb{F}_2{}^4$, and therefore

$$Aut(Z_2{}^4) = GL_4(2), \text{ with order } (16-1)(16-2)(16-4)(16-8) = 20160 = 8!/2. \quad (6.2)$$

The groups $GL_4(2)$ and Alt_8 are isomorphic [2]; but notice also other numerical coincidence:

$$|PSL_3(4)| = (64-1)(64-4)(64-16)/3/3 = 20160, \quad (6.3)$$

but these groups are *not* isomorphic: 20160 is the smallest number for the order of two non-isomorphic *simple* groups (Artin). This anticoincidence is also related to the "large" Mathieu groups M_{22}, M_{23} and M_{24}[†].

It is remarkable the increase in automorphisms as one goes from the first $A = Z_{16}$ to the fifth, the elementary abelian $A = (Z_2)^4$: the orders of $(AutA)$ are 8, 16, 96, 192 and 20160 respectively: this is reflected also in the partition by numbers: calling $n_{1,2,4,8,16}$ the number of elements of this order, for 16/1... up to 16/5:

$$n_{1,2,4,8,16} : (1,1,2,4,8).-(1,3,4,8,0).-(1,3,12,0,0).-(1,7,8,0,0).-(1,15,0,0,0). \quad (6.4)$$

7 Non-abelian groups of order 16

See [32] as a modern reference, therefore we shall report on them rather briefly. We start by dividing the groups in *three* families:

First family: (Semi-) Direct products; $D_4 \times Z_2$, $Q \times Z_2$ and $Q \rtimes Z_2$

$$16/6\ D_4 \times Z_2. - \quad 2(4 \cdot 1^2 + 1 \cdot 2^2) = 8 \cdot 1^2 + 2 \cdot 2^2 : \quad (7.1)$$

There are then TEN classes, and the class split is deduced from the structure $(Z_4 \rtimes Z_2) \times Z_2 : (a, \alpha, b; a^4 = a\alpha^2 = b^2 = e; \alpha a \alpha = a^3; ab = ba, \alpha b = b\alpha)$.

$$n_{1,2,4,8,16} = 1, 11, 4, 0, 0 \qquad \text{with} \quad 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 4 \cdot 2_2 + 2 \cdot 2_4 \quad (7.2)$$

There are plenty of Out elements; notice $Centre = V$, $Ab = Z_2^3$, and $Inn = V$. One shows $|Aut| = 64$ (as a can go to 4 places, α to 8, and then b to 2), hence $|Out| = 16$.

$$16/7 \qquad Q \times Z_2 \qquad \text{with, as above,} \quad 16 = 8 \cdot 1^2 + 2 \cdot 2^2, \text{ and}$$
$$n_{1,2,4,8,16} = 1, 3, 12, 0, 0 \qquad \text{with} \quad 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 6 \cdot 2_4 \quad (7.3)$$

[†](L.J. Boya, Aug-2010). Contribution to the ICM in Hyderabad, India. Unpublished.

As before, $G' = Z_2$, $Z(G) = V$, $Inn(G) = V$, one shows $|Aut| = 192$, hence $|Out| = 48$.

16/8 $Q \rtimes Z_2$: as $Aut(Q) = S_4$ and $Out(Q) = S_3$, $Z_2 \subset S_3 \subset S_4$ and the semidirect product makes sense.

Second family: Dihedral groups, 16/9 to 16/11: D_8 , D_8' , and D_8''

$$\underline{16/9} \quad D_8 = Z_8 \rtimes Z_2 : \quad \{a^8 = \alpha^2 = e, \alpha \cdot a \cdot \alpha = a^7 = a^{-1}\}, 16 = 4 \cdot 1^2 + 3 \cdot 2^2,$$
$$\text{Dihedral group} \quad n_{1,2,4,8,16} = 1, 9, 12, 4, 0 \text{ with } 16 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 2_4 + 1 \cdot 2_4 + 2 \cdot 2_8 \quad (7.4)$$

Center is Z_2, G_{ab} is Z_4, Out is $Z_2 \times Z_2 = V$; so $|Aut| = (16/2) \cdot 4 = 32$; one shows also $Aut = Hol(Z_8) = Z_8 \rtimes V$. D_8 is the full Symmetry group of the regular octagon.

As we anticipated, due to the fact that $Aut(Z_8) = V$, there are two more involutive autos, which allow for two more (different) "dihedral-type" groups:

$$\underline{16/10} \qquad D_8' = Z_8 \rtimes Z_2 : \qquad \{a^8 = \alpha^2 = e, \alpha \cdot a \cdot \alpha = a^3\}, 16 = 4 \cdot 1^2 + 3 \cdot 2^2,$$
$$n_{1,2,4,8,16} = 1, 5, 6, 4, 0 \text{ with } 16 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 4_2 + 1 \cdot 2_4 + 2 \cdot 4_4 + 2 \cdot 2_8 \quad (7.5)$$

Center Z_2 implies $|Inn| = 8$, and as obviously $|Out| = 2$, we have $|Aut| = 16$; indeed $Aut(D_8') = Q \rtimes Z_2$.

Now, as when $\alpha \cdot a \cdot \alpha = a^5$, the a^2 and a^6 remain fixed, things are a bit different in D_8'':

$$\underline{16/11}\ (TW : 16/11)\ D_8'' = Z_8 \rtimes Z_2 : \qquad \{a^8 = \alpha^2 = e, \alpha \cdot a \cdot \alpha = a^5\}, 16 = 4 \cdot 1^2 + 2 \cdot 2^2,$$
$$n_{1,2,4,8,16} = 1, 3, 4, 8, 0 \text{ with } 16 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 2_2 + 2 \cdot 1_4 + 1 \cdot 4_2 + 4 \cdot 2_8$$

Third family: "4×4" groups, 16/12 to 16/14:

16/12: $Z_4 \rtimes Z_4$: the auto is defined as $Aut(Z_4) = Z_2$, and $Z_4 \to Z_2$. We have

$$16 = 8 \cdot 1^2 + 2 \cdot 2^2, \qquad \text{ten classes, and } 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 6 \cdot 2_4 \qquad (7.6)$$

Center is V, G_{ab} is $Z_4 \times Z_2$, and Inn is V; hence, $|Aut| = 32$, as $|Out| = 8$.

16/13: $V \rtimes Z_4 = (Z_2 \times Z_2) \rtimes Z_2$: the auto is an involution within $Aut(V) = S_3$:

$$16 = 8 \cdot 1^2 + 2 \cdot 2^2. - \qquad n_{1,2,4,8,16} = (1, 7, 8, 0, 0) \quad \text{with } 16 = 1 \cdot 1_1 + 3 \cdot 1_2 + 2 \cdot 2_2 + 4 \cdot 2_4 \quad (7.7)$$

One has $|Out| = 8$; as $Inn = V$, $|Aut| = 32$.

16/14: Finally, we have the Dicyclic $Q_4 = Z_8 \rtimes_{/2} Z_4$, $16 = 4 \cdot 1^2 + 3 \cdot 2^2$, 7 classes $\{a^8 = \alpha^4 = e, a^4 = \alpha^2, \alpha \cdot a \cdot \alpha^{-1} = a^{-1}\}$

$$n_{1,2,4,8,16} = (1, 1, 10, 4, 0) \qquad \text{with } 16 = 1 \cdot 1_1 + 1 \cdot 1_2 + 1 \cdot 2_4 + 2 \cdot 4_4 + 2 \cdot 2_8 \qquad (7.8)$$

We have easily $Center = Z_2$, $G_{ab} = V$, $Inn = D_4$, $|Aut| = 32$ and $|Out| = 4$

8 Groups with order $16 < |G| < 32$, order p or pq

Consider first $|G| = p$ or pq (one or two prime factors *only*). That means in this level orders $17, 19, 23, 29, 31$ (prime p) and orders $21, 22, 25, 26$ (two primes pq).

For the first case, prime order, the only group is the corresponding Z_p:
$\underline{17/1, 19/1, 23/1, 29/1}$ and $\underline{31/1}$. e.g. for $p = 29$ as the example,

$\underline{29/1}$ Z_{29}, abelian, cyclic and simple; $29 = 29 \cdot 1^2$; $29 = 1 \cdot 1_1 + 28 \cdot 1_{29}$. $Aut(Z_{29}) = Z_{28} = Z_7 \times Z_4$; rotation symmetry of the 29-th regular polygon.

Now for the case of two prime factors:

Groups of order 21. As anticipated, there are two, as 3 and 7 are *compatible* primes (i.e. $(7 - 1)$ divides 3):

$\underline{21/1}$, namely $Z_{21} = Z_7 \times Z_3 :$ $21 = 1 \cdot 1_1 + 6 \cdot 1_3 + 2 \cdot 1_7 + 12 \cdot 1_{21}$, $Aut(Z_{21}) = Z_6 \times Z_2 = V \times Z_3 = \underline{12/2}$.

$\underline{21/2}$: G_{21}, called sometimes Frobenius group, $G_{21} = Z_7 \rtimes Z_3$: this group acts transitively in the so-called Fano plane, related to the octonion numbers... [9]

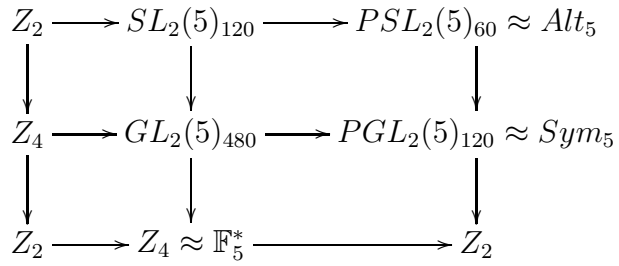
$$21 = 3 \cdot 1^2 + 2 \cdot 3^2, \qquad 21 = 1 \cdot 1_1 + 2 \cdot 7_3 + 2 \cdot 3_7. \qquad (8.1)$$

For order 22, we have the cyclic and the dihedral, as for order 10, 14, etc.:
$\underline{22/1}$: $Z_{22} = Z_2 \times Z_{11}$.

$\underline{22/2}$: The dihedral $D_{11} = Z_{11} \rtimes Z_2$.

For order $25 = 5^2$ we have the two abelian solutions, as $Part(2) = 2$.
$\underline{25/1}$: Z_{25}, cyclic. $Aut(Z_{25}) = Z_{25-5=20}$.

$\underline{25/2}$: $(Z_5 \times Z_5)$ (elementary abelian: ${\mathbb{F}_5}^2$, $Aut = GL_2(5)$, order $480 = (5^2 - 1)(5^2 - 5)$).
It is good to exhibit the full structure of the $GL_2(5)$ group:



Notice the isomorphisms: $PSL_2(5) \approx Alt_5$, the first nonabelian simple group; and $PGL_2(5) \approx Sym_5$: the group $PGL_2(5)$ is sharp 3-transitive in 5+1 symbols (Cfr. App. A), whereas Sym_5 is sharp 5-*trans* in 5, but this is the same number, as $6{\cdot}5{\cdot}4 = 5{\cdot}4{\cdot}3{\cdot}2{\cdot}1$. The groups are also isomorphic: e.g. Sym_5 has elements of order 6.

As for order 26 is:

26/1: $Z_{26} = Z_2 \times Z_{13}$. $Aut = I \times Aut(Z_{13}) = Z_{12}$.

26/2: $D_{13} = \mathrm{Z}_{13} \rtimes Z_2$. $Aut(\mathrm{D}_{13}) = Z_6$, as $|Hol(Z_{13})| = 13{\cdot}12$. $26 = 1{\cdot}1_1(e){+}6{\cdot}2_{13}{+}1{\cdot}13_2$

9 Groups of order 24

As $24 = 2^3 \cdot 3$, there are three abelian groups: (24/1,2,3)

$$Z_{24} = Z_8 \times Z_3, \quad Z_4 \times Z_2 \times Z_3 = Z_6 \times Z_4, \quad {Z_2}^3 \times Z_3 = Z_6 \times V. \tag{9.1}$$

As examples, the class-order partition for the three are

$$24 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 1_3 + 2 \cdot 1_4 + 2 \cdot 1_6 + 4 \cdot 1_8 + 4 \cdot 1_{12} + 8 \cdot 1_{24} \qquad (\text{for } Z_{24})$$
$$24 = 1 \cdot 1_1 + 7 \cdot 1_2 + 2 \cdot 1_3 + 14 \cdot 1_6 \qquad (\text{for } Z_6 \times Z_4)$$
$$24 = 1 \cdot 1_1 + 7 \cdot 1_2 + 8 \cdot 1_3 + 8 \cdot 1_6 \qquad (\text{for } Z_6 \times V)$$

As for the *Aut* groups, we have $Aut(Z_{24}) = Aut({Z_2}^3) \times Aut(Z_3) = Aut({Z_2}^3) \times Z_2$, so

$$Aut(Z_{24}) = Aut(Z_8) \times Aut(Z_3) = V \times Z_2 = (Z_2)^3,$$
$$Aut(Z_4 \times Z_2 \times Z_3) = D_4 \times Z_2, \qquad \text{and}$$
$$Aut({Z_2}^3 \times Z_3) = GL_3(2) \times Z_2, \text{ order } 168 \cdot 2 = 336.$$

As for the non-abelian groups of order 24, several at once are direct products:

24/4: $D_6 \times Z_2 = (Z_6 \rtimes Z_2) \times Z_2 = Z_2 \times (Z_3 \rtimes Z_2) \times Z_2 = S_3 \times V$.

24/5: $Alt_4 \times Z_2 = (V \rtimes Z_3) \times Z_2$

24/6: $\mathrm{Q}_3 \times Z_2 = (Z_3 \rtimes Z_4) \times Z_2$.

24/7: $\mathrm{D}_4 \times Z_3$.

24/8: $\mathrm{Q} \times Z_3$.

24/9: $\mathrm{S}_3 \times Z_4$. (Notice $\mathrm{S}_3 \times V =$ 24/4: easy to prove).

As 24 divisible by 2 and 4, and also $24 = 4!$, three more groups are immediate;
24/10: $\mathrm{D}_{12} = (Z_{12} \rtimes Z_2) = (Z_3 \times Z_4) \rtimes Z_2$.

24/11: $\mathrm{Q}_6 := Z_6 \rtimes\!/_2 Z_4$.

24/12: $\mathrm{S}_4 \equiv \mathrm{Sym}_4 = Alt_4 \cdot 2 = Alt_4 \rtimes Z_2$.

As the last (Sym_4) is one of the few cases of semidirect product with nonabelian kernel, we specify why is it possible: The *outo* in Alt_4 mixing the two (123)-type classes is involutive. But it can be realized also as an *involutive auto*, because, for example, conjugation of a (123) permutation by an involution, e.g. (12) does generate the other 3-cycle class! Hence, one writes correctly $Sym_4 = Alt_4 \rtimes Z_2$.

Finally, we shall show later (last, Sect. 12) that the Alt_n groups have a $2 \cdot Alt_n$ extension (Schur *multiplier*), so our next group is
24/13: $SL_2(\mathbb{F}_3) \approx 2 \cdot Alt_4$.

For more detailed information, see [29]. Notice only the diagram (Cfr. 8.1)

$$\begin{array}{ccc} 2 \cdot Alt_4 \approx SL_2(3)_{24} & \longrightarrow & PSL_2(3)_{12} \approx Alt_4 \\ \downarrow & & \downarrow \\ GL_2(3)_{48} & \longrightarrow & PGL_2(3)_{24} \approx Sym_4 = Alt_4 \cdot 2 \end{array} \tag{9.2}$$

Finally, in the product $24 = 3 \cdot 8$ the group Z_3 admits semidirect extensions as in
24/14: $Z_3 \rtimes Z_8$ (where Z_8 acts in $Z_2 = Aut(Z_3)$ by $Z_8 \to Z_2\ mod\, Z_4$).

24/15: $Z_3 \rtimes D_4$ (where similarly D_4 acts in $Z_2 = Aut(Z_3)\ mod\, V$).

This completes our study of order 24 groups. There are $15 = 3 + 12$ of them.

10 Rest of groups

$16 < |G| < 32$. $|G| = pqr$: 18, 20, 27, 28 and 30 ($|G| \neq 24$).

Order 18: Two abelians, as $18 = 2 \cdot 3^2$ and three no-abelians: D_9, $\mathrm{S}_3 \times Z_3$ and $({Z_3}^2) \rtimes Z_2$. For the last one, note $Aut({Z_3}^2) = GL_2(3) \supset Z_2$. The description is straightforward; we omit to make some of the *autos* explicit, for simplicity. Notice the Burnside possibility $18 = 9 \cdot 1^2 + 1 \cdot 3^2$ is not realized. Notice also $Hol(Z_9) = Z_9 \rtimes Z_6 \supset Z_9 \rtimes Z_2$.

18/1: $Z_{18} = Z_9 \times Z_2$. $18 = 1{\cdot}1_1 + 1{\cdot}1_2 + 2{\cdot}1_3 + 2{\cdot}1_6 + 6{\cdot}1_9 + 6{\cdot}1_{18}$; $Aut(Z_{18}) = Z_6$

18/2: $(Z_3)^2 \times Z_2 = Z_6 \times Z_3$. $18 = 1{\cdot}1_1 + 1{\cdot}1_2 + 8{\cdot}1_3 + 8{\cdot}1_6$ $Aut = GL_2(3)$

18/3: for $S_3 \times Z_3$, $18 = 6{\cdot}1^2 + 3{\cdot}2^2$; $18 = 1{\cdot}1_1 + 1{\cdot}3_2 + 1{\cdot}2_3 + 2{\cdot}1_3 + 2{\cdot}2_3 + 2{\cdot}3_6$ $Out = Z_2$.

18/4: $D_9 := Z_9 \rtimes Z_2$; $18 = 2 \cdot 1^2 + 4 \cdot 2^2$; $18 = 1 \cdot 1_1 + 1 \cdot 9_2 + 1 \cdot 2_3 + 3 \cdot 2_9$.

18/5: for $(Z_3)^2 \rtimes Z_2$; $18 = 2 \cdot 1^2 + 4 \cdot 2^2$; $18 = 1 \cdot 1_1 + 1 \cdot 9_3 + 4 \cdot 2_2$.

For order 20 the situation is very similar: $20 = 2^2 \cdot 3$, so 2 Abelian plus 3 non: only the structure is given, but for 20/5:

20/1: $Z_{20} = Z_5 \times Z_4$.

20/2: $Z_5 \times (Z_2)^2 = Z_5 \times V$.

20/3: $D_{10} = Z_{10} \rtimes Z_2 = (Z_5 \times Z_2) \rtimes Z_2 = (Z_5 \rtimes Z_2) \times Z_2 = D_5 \times Z_2$ $20 = 4 \cdot 1^2 + 4 \cdot 2^2$.

20/4: Q_5 Dicyclic $= Z_{10} \rtimes|_2 Z_4 = Z_5 \rtimes V$ $20 = 8 \cdot 1^2 + 3 \cdot 2^2$.

20/5: $Z_5 \rtimes Z_4 = Hol(Z_5)$: $20 = 4 \cdot 1^2 + 1 \cdot 4^2$, $20 = 1 \cdot 1_1 + 1 \cdot 5_2 + 2 \cdot 5_4 + 1 \cdot 4_5$.

For order $27 = 3^3$, we have three abelian groups, as $Part(3) = 3$ and two nonabelian, as for order 8:

27/1: Z_{27}, cyclic; $27 = 27 \cdot 1^2$; $27 = 1 \cdot 1_1 + 2 \cdot 1_3 + 6 \cdot 1_9 + 18 \cdot 1_{27}$, $|Aut(Z_{27}| = 9$.

27/2: $Z_9 \times Z_3$, $27 = 1 \cdot 1_1 + 8 \cdot 1_3 + 18 \cdot 1_9$.

27/3: $(Z_3)^3$, elementary abelian. $27 = 1 \cdot 1_1 + 26 \cdot 1_3$, $|Aut((Z_3)^3)| = |GL_3(3)| = 11232$.

As $Aut(Z_9) = Z_6$ and $Aut({Z_3}^2) = GL_2(3)$, $|\cdot| = 48$, there is map $Z_3 \to Aut({Z_3}^2)$. Hence, the two nonabelian are semidirect products (this differs from the order 8 case):

27/4: $({Z_3}^2) \rtimes Z_3$: $27 = 9 \cdot 1^2 + 2 \cdot 3^2$, and $27 = 1 \cdot 1_1 + 2 \cdot 1_3 + 8 \cdot 3_2$.

27/5: $Z_9 \rtimes Z_3$: $27 = 9 \cdot 1^2 + 2 \cdot 3^3$, and $27 = 1 \cdot 1_1 + 2 \cdot 1_3 + 2 \cdot 3_3 + 18 \cdot 3_9$.

As for the case of order 8, these two order 27 groups are *extraspecial*, as the center is Z_3 and the abelianized $(Z_3)^2$.

Notice 27/2 and 27/5 have the same partition by orders (namely, $n_3, n_9 = 8, 18$).

In [9] groups with this property are called *conformal*; note they are *not* isomorphic.

For order 28, there are only *four* groups: 2 Ab + 2 non:

28/1: $Z_{28} = Z_7 \times Z_4$ $Aut = Z_6 \times Z_2$.

28/2: $Z_7 \times Z_2 \times Z_2 = Z_7 \times V = Z_{14} \times Z_2 \quad Aut = Z_6 \times S_3$.

28/3: $\mathrm{D}_{14} := Z_{14} \rtimes Z_2 = \mathrm{D}_7 \times Z_2 \quad Out = Z_3$.

28/4: $Z_7 \rtimes Z_4 = Q_7 = Z_{14} \rtimes_2 Z_4, \quad 28 = 4{\cdot}1^2{+}6{\cdot}2^2 = 1{\cdot}1_1{+}1{\cdot}1_2{+}2{\cdot}7_4{+}3{\cdot}2_7{+}3{\cdot}2_{14} \quad Out = Z_6$.

One might ask: why in orders $12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$ and $20 = 2^2 \cdot 5$ are there five groups, whereas in $28 = 2^2 \cdot 7$ there are only four? The answer is: 12, 18 and 20 are special! The first has $V \rtimes Z_3 = Alt_3$ as special, the second $S_3 \times Z_3$ is special, and the third $Z_5 \rtimes Z_4 = Hol(Z_5)$, also special.

Order 30 is interesting, as there is only an abelian group, inspite of having three factors, but none is repeated: $30 = 2 \cdot 3 \cdot 5$. Hence
30/1: $Z_{30} = Z_2 \times Z_3 \times Z_5 = Z_6 \times Z_5 = Z_{10} \times Z_3 = Z_2 \times Z_{15}$, etc. $30 = 1 \cdot 1_1 + 1 \cdot 1_2 + 2 \cdot 1_3 + 4 \cdot 1_5 + 4 \cdot 1_{10} + 8 \cdot 1_{15} + 8 \cdot 1_{30}$. $Aut(Z_{30}) = Z_4 \times Z_2$.

and there are three nonabelian as 2 is compatible with any prime, but 3 and 5 are not:

30/2: $\mathrm{D}_5 \times Z_3, \quad 30 = 6{\cdot}1^2{+}6{\cdot}2^2, \quad 30 = 1{\cdot}1_1{+}1{\cdot}5_2{+}2{\cdot}1_3{+}2{\cdot}2_5{+}2{\cdot}5_6{+}4{\cdot}2_{15} \quad Out = V$.

30/3: $(Z_3 \rtimes Z_2) \times Z_5 = S_3 \times Z_5, \quad 30 = 1{\cdot}1_1{+}1{\cdot}3_2{+}1{\cdot}2_3{+}4{\cdot}1_5{+}4{\cdot}3_{10}{+}4{\cdot}2_{15} \quad Out = Z_4$.

30/4: $\mathrm{D}_{15} = Z_{15} \rtimes Z_2, \quad 30 = 2{\cdot}1^2{+}7{\cdot}2^2, \quad 30 = 1{\cdot}1_1{+}1{\cdot}15_2{+}1{\cdot}2_3{+}2{\cdot}2_5{+}4{\cdot}2_{15} \quad Out = Z_4$.

This completes our study of the $48 + 45 = 93$ groups G, with $|G| < 31$.

11 Tables

In this Section we just group the found groups in some classes, for completeness.

Table 3 recalls the total counting, separating the abelian ones.

	Abelian	Non-abelian	Total	
$\|G\| < 8$	8	1	9	
$\|G\| = 8$	3	2	5	
$8 < \|G\| < 16, (\neq 12)$	7	2	9	
$\|G\| = 12$	2	3	5	
				$\overline{20+8}$
$\|G\| = 16$	5	9	14	
$16 < \|G\| < 24$	9	8	17	
$\|G\| = 24$	3	12	15	
$24 < \|G\| \leq 31$	11	8	19	
	48	45	$\underline{93}$	

Table 3.—

Table 4 selects some Abelian groups with their *Aut* group, or $|Aut|$ order.

Group	Order	Struct.	*Aut*	$\|Aut\|$	Comments
V	4	${Z_2}^2$	S_3	6	First non-cyclic group.
Z_8	8	cyclic	V	4	Cyclic, *Aut* group not.
${Z_2}^3$	8	${\mathbb{F}_2}^3$	$GL_3(2)$	168	(2^{nd} smallest nonabel. simple)
Z_{24}	24	$Z_8 \times Z_3$	$V \times Z_2$	8	
Z_{30}	30	$Z_2 \times Z_3 \times Z_5$	$Z_2 \times Z_4$	8	

Table 4.— Some interesting abelian groups, $|A| < 32$.

Group	Order	Struct.	Out	$\lvert Out\rvert$	Comments
S_3	6	$Z_3 \rtimes Z_2$	e	1	Smallest non-abelian.
Q	8	Quaternion	S_3	6	Smallest, no $\times$ nor $\rtimes$.
Alt_4	12	$V \rtimes Z_3$	Z_2	2	$Alt_4 \cdot 2 = Sym_4 = Alt_4 \rtimes Z_2$.
Dih'_8	16	$Z_8 \rtimes Z_2$			$\alpha a \alpha = a^5$.

Table 5.— Some interesting non-abelian groups, $|G| < 32$.

Group	Order	Aut	Hol	$\lvert Hol\rvert$
Z_3	3	Z_2	S_3	6
$V = {\mathbb{F}_2}^2$	4	S_3	S_4	24
${\mathbb{F}_2}^3$	8	$GL_3(2)$	$Aff_3(2)$	1344
${\mathbb{F}_3}^2$	9	$GL_2(3)$	$Aff_2(3)$	48

Table 6.— Some interesting Holomorphs of certain groups, $|G| < 32$.

12 Final comments

Here we comment some isolated topics not deal with in the main text.

1. Extraspecial groups. For order $8 = 2^3$ and $27 = 3^3$ the nonabelian groups (two in each case) are *extraspecial*, as we noted. This is reflected in the center, being Z_2 and Z_3 respectively, and the abelianized, being $(Z_2)^2$ and $(Z_3)^2$ respectively. One shows [14] they all come from $(Z_p)^3$, p prime, where the two nonabelian groups are the extraspecial.

2. Schur multipliers. (To understand the following some knowledge of the mathematics of Quantum Theory is necessary; see e.g. [30] or [23]).

   The transition from integer spin "ℓ"$= 0, 1, 2, 3, ...$ in orbital angular momentum to "half-integer" for spin particles in Quantum Mechanics (QM), so $s = 0, 1/2, 1, 3/2, 2, ...$ is due to the need of using *projective* representations of the $SO(3)$ group, because in QM states are rays, not vectors. The change from $SO(3)$ to $Spin(3)$ is, as an extension problem, a double (universal) covering;

   $$Z_2 \longrightarrow SU(2) = Spin(3) \longrightarrow SO(3) \tag{12.1}$$

   and we wrote in several ocassions $SU(2) = 2 \cdot SO(3)$: this 2 "on the left" is a case of Schur multiplier; in general, extensions of type

   $$Z_n \longrightarrow G\hat{} \longrightarrow G \tag{12.2}$$

are called "Schur multipliers", because I. Schur was the first to consider projective representations of groups (ca. 1898).

The complete story of this goes through the extension theory, which requires a study of *Homological Algebra*, that we just omit.

3. The attentive reader has probably observed that the majority of our non-abelian groups are:

   - Simple, if prime order, Z_p.
   - Direct product, whenever possible; e.g. 28/3: $D_7 \times Z_2$.-
   - Semidirect product, e.g. $S_3 = Z_3 \rtimes Z_2$.

   In very few occasions, we have quotients of semidirect products, e.g. in order 8, $Q = Q_2 \equiv Z_4 \times /_2 Z_4$. Why non-simple groups are nearly always direct or semidirect products? The reason is [6] that, in the general extension problem, that we just mention in Appendix A, for orders of G with small factors, the group of the extensions reduces to the unity. And precisely the cases non of this type start with three or more factors identical, that is to say, 8, 16, 24, etc., where indeed there are groups not in simple, direct or semidirect form.

A APPENDIX

Here we collect some definitions and show easy results in the abstract theory of finite groups; general references are ([25], [17], [14], and [28]). All groups will be finite, order of $G := |G| = n < \infty$; three other classical sources for finite groups are [9], [8] and [20]. The unity will be named e or Id or id; for I we signal at times the group only with $\{e\}$. For a group G the *opposed* group, named G^o has the same elements as G, but to the pair (a, b) corresponds the product (ba). ab and ba are in the same class, as $a^{-1}aba = ba$. G is *abelian* iff $G = G^o$. Groups and their *natural* maps (homomorphisms) form a *category* $\mathcal{G}$, and finite groups the (sub)category $\mathcal{G}^{\circ\circ}$; but we shall not say much of that.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will be the natural, integer, rational, real and complex numbers, as usual; quaternions $\mathbb{H}$ will be mentioned very sporadically. For any natural number n there always exists a *cyclic* group (and only one), named Z_n, where $Z_n : \{a \mid a^n = e\}$; it is also written Z/nZ, read integers with sum $mod\, n$ (Gauss). For a set X with n points, the *permutation group* or collection of all permutations under composition, has order $n!$; it is written Sym_n or simply S_n; the *even* permutations make a normal subgroup, called alternating group Alt_n, with $n!/2$ elements.

1. General definitions. In a set X we give a *composition law* $X \times X \to X$, which is *associative* $x(yz) = (xy)z$, with *unit* e $(xe = ex = x)$ and *inverse* x^{-1} $(xx^{-1} = x^{-1}x = e)$: this is the abstract definition of a group (A. Cayley, 1854); as said we consider only finite groups G, $|G| < \infty$. Then the *period* of an element is the minimum n such $a^n = e$.

   Given a group G, a *subgroup* H is a subset of G which is group by itself; G splits under H in left gH (or right - Hg) *cosets*, so e.g. $G = H \cup gH \cup g'H \cup g''H...$ Each *coset* gH has order $|H|$, because gH with $g \notin H$ just changes k in H by gk: this is *Lagrange theorem*: the order $h = |H|$ of a subgroup $H \subset G$ divides the order of G, $n = |G|$; one writes $[G : H]$ for the *index* of H in G, so $[G : H] = n/h$.

   The "left translation" $g : g' \to gg'$ permutes the elements of the set G; hence the group itself G, $|G| = n$, can be seen as a subgroup of the general permutation group S_n: this is *Cayley theorem*. $H \subset G$ is *normal* subgroup (invariant, distinguished) if $gHg^{-1} = H \Leftrightarrow ghg^{-1} = h'$. If H normal in G, there is a group structure in the cosets, by composition, called the *quotient* group Q; one writes $G/H = Q$. We shall write this as

$$1 \to H \to G \to Q \to 1 \tag{A.1}$$

(*short exact sequence*), meaning: the second map is injective, the third epijective, and composition of any two consecutive maps gives 1. Normally we shall omit the two "1"s.

If $|G| = p^f \cdot m$, with m natural, p prime and f natural, with $(m, p) = 1$, there are in G subgroups of any order p^r, $1 \leq r \leq f$; the subgroups of maximal order p^f are called *Sylow subgroups*, all are conjugate, and the number of them is $1+kp$, dividing $|G|$ (*Sylow theorems*); see [11]. There is a *partial* ordering of the subgroups H of a group G by inclusion; one thus forms the lattice of subgroups.

Natural maps $\mu : G \to Q$ are *homomorphisms*, i.e. when $\mu(gh) = \mu(g)\mu(h)$. The image of μ is the subgroup $\mu(G) \subset Q$, and μ is *onto* when $\mu(G) = Q$. (Homo-)Morphisms $\mu : G \to G$ are called endomorphisms (*endos*), and isomorphisms ι are invertible morphisms, as automorphisms α (*autos*) are invertible *endos*. In any morphism $\mu : G \to Q$ the kernel $K := \mu^{-1}(e_Q)$ is *normal* subgroup, $gKg^{-1} = K$, as $k \in K \Rightarrow \mu(gkg^{-1}) = \mu(g)\mu(k)\mu(g^{-1}) = \mu(g)\mu(g^{-1}) = e$. M is *injective* if $ker\, K = \{e\}$. One writes $G/K = Im\, \mu = \mu(G) \subset Q$: the image is just subgroup. If two groups G_1 and G_2 are isomorphic, we write $G_1 \approx G_2$: we try to reserve $G_1 = G_2$ for two groups *defined* identically. For example $Z_6 \approx Z_2 \times Z_3$, but ${Z_2}^2 = V$.

One writes a general morphism $\mu : G \to Q$ as composition of $Ker\, \mu \to G \to Im\, \mu$ and injection $Im\, \mu \to Q$.

The set of *autos* of any algebraic (or geometrical) structure is always a group under composition. This is the *very reason* of the existence of Groups as algebraic structures. In our study $Aut(G)$ will play an important role; of course, $Aut(G)$ is finite if G is. But A can be abelian and $Aut(A)$ not; or Z being infinite, has $Aut(Z)$ finite (± 1).

For abelian groups A one sets up a *ring* structure in the whole set of *endos*, $End(A)$, by defining $(a + b)(x) := a(x) + b(x)$, besides $(ab)(x) = a(b(x))$; the *autos* are the invertible elements in $End(A)$; (so $0 : a \to e$ is *never* an auto, but $1 : a \to a$ is *always*). Recall a field is a *commutative* ring with all elements $\neq 0$ having inverse under product.

Conjugation of element g by another element k is the map $i_k : g \to kgk^{-1}$, and it is an automorphism in G, called *inner*; their set is named $Inn(G)$, with $Inn(G) \subset Aut(G)$, as (normal) subgroup. The above *onto* map $G \to Inn(G)$ is morphism, with kernel the center (centre) of G, namely $Z(G) : \{z | zg = gz\}$. An automorphism which is not inner is called *outer*: the quotient $Out(G) := Aut/Inn$ is called the *group of classes* of outer automorphisms. An element g and all its conjugates kgk^{-1} make up the class of $g, cl(g)$.

Automorphisms $\alpha \in Aut(G)$ keep the order, as $a^n = e \Rightarrow \alpha(a)^n = e$ with n period; hence, in order a group to have automorphisms $\neq e$, *it has to have more than one element with the same period*; for example, $Aut(Z_2) = I$. If G is nonabelian, the $Out(G)$ group operates in the classes (as orbits under conjugation, i. e. inner automorphisms); hence, for having *outos* $\gamma \neq e$, *a group* G *has to have classes of the same periods (many examples).*

The commutator of two elements a, b is $(a, b) := aba^{-1}b^{-1}$; it is $(,) = e$ iff $ab = ba$; the clausure of commutators makes up another *normal* subgroup, called the *derived* (or commutator) subgroup $G' = \{aba^{-1}b^{-1}\}$. The quotient $G/G' := G_{ab}$ is abelian, and $G \to G_{ab}$ is the *maximal* abelian image of G, with $A = G_{ab}$ iff G abelian already. For any subset X of a group G, the *centralizer* in G, $Z_G(X)$ is the subgroup $\{g|gx = xg, x \in X\}$; the *normalizer* $N_G(X)$ is defined by $\{g|gxg^{-1} = x' \in X\}$; by $\langle X \rangle$ we mean the group *generated* by elements in X, that is, the minimum group containing the set X.

For example, the centralizer of G itself is the center, $Z(G)$; the normalizer of a normal subgroup H is the whole group, G; etc.

We express these definitions and results by the following diagram, very much used (implictly) in this review

$$\begin{array}{ccccc} & & Z(G) & & \\ & & \downarrow & & \\ G' & \longrightarrow & G & \longrightarrow & G_{ab} \\ & & \downarrow & & \\ & & Inn(G) & \longrightarrow Aut(G) \longrightarrow & Out(G) \end{array}$$

A normal subgroup is invariant under inner *autos*, by definition; if it is invariant under *any* automorphism, it is called *characteristic* subgroup; one shows easily that the center $Z(G)$ is characteristic, and so is the commutator G'. Therefore, there is a map $Out(G) \to Aut(Z(G))$ as $Inn(G)$ operates trivially in the center.

A group is simple if the only normal subgroups are the trivial ones, namely I and the group G itself. For example, Z_p, p a prime number, is simple; so is $Alt_{n>4}$ (Galois). The smallest non-abelian simple group is Alt_5, order $5!/2 = 60$, and the second smallest is $PSL_2(7) = GL_3(2)$, order 168.

A group is called *complete* if $Center = Out = e$; *perfect* if $G = G'$ (so $G_{ab} = e$). For example, S_3 is complete, and Alt_5 is perfect; perfect groups, if not simple, are *rare*: about the smaller *nonsimple* perfect group is $SL_2(5)$, order 120; *note $Alt_5 \approx PSL_2(5)$*.

We define the *class* $[g] \equiv cl(g)$ of an element as the set of its conjugate elements: $[g] = \{k, k = hgh^{-1}\}$; then "belonging to a class" is an *equivalence relation* in G, and so one writes $G = \cup_{suff\ classes}\,[g]$; for example, for the cyclic group Z_p, p a prime number, one has $|Z_p| = p$ and there are one class of 1 element of order $1(e)$, plus $(p-1)$ classes of 1 element, of order p; we write this as $p = 1 \cdot 1_1(e) + (p-1) \cdot 1_p(a, a^2, a^3, ..., a^{p-1})$.This partition by classes is the same as the orbits of the group acting on itself by conjugation. For any finite group G, one can write, in general

$$|G| := N = \sum n \cdot m_s, \tag{A.2}$$

where $n \cdot m_s$ means: n classes of m elements each, of period s. Elements in the same class have same period, but of course not viceversa: a *more rough* classification of elements is by counting $\{n_s\}$, the number of elements for each period. For example, $\{n_{s=1,2,4,8,16}\} = (1, 1, 2, 4, 8)$ in Z_{16}. In [8] one defines two groups to be *conformal* if the partition by classes (A.2) is the same.

With two groups $G = \{g\}$ and $K = \{k\}$ we form the *direct product* in the product *set* $G \times K$ by $(g, k) \cdot (g', k') := (gg', kk')$. For example, one shows thus that $Z_6 = Z_2 \times Z_3$, $Z_{15} = Z_3 \times Z_5$, etc. This is the simplest case of the extension problem: given two groups, K for kernel and Q for quotient, form all groups E with K normal and $E/K = Q$. The complete solution of this question requires *Homological Algebra*, which we shall not develop.

If A is abelian (with $a + b$ for ab) and there is a morphism $\mu : G \to Aut(A)$, in the set $A \times G$ one forms a group with the composition $(a, k) \cdot (a', k') = (a + \mu_k(a'), kk')$, and it is called the *semidirect* product, written $A \rtimes G$. For example, one shows $S_3 = Z_3 \rtimes Z_2$. We shall use a lot this semidirect product construction. There is also an extension of the semidirect product for non-abelian groups, $B \neq B^o$, which we shall use far less.

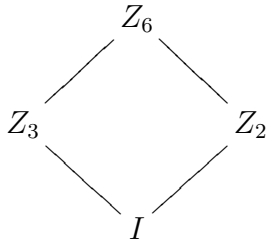
As an example of (A.2), for the permutation group Sym_4, of order $4! = 24$, there are 5 classes, as $Part(4) = 5$; the reader should check the balance (we write one representative per class)

$$4! = 24 = 1 \cdot 1_1(e) + 1 \cdot 6_2([12]) + 1 \cdot 3_2([12;34]) + 1 \cdot 8_3[123] + 1 \cdot 6_4([1234] \tag{A.3}$$

where $[12]$ is the transposition $1 \to 2 \to 1$, etc: we suppose the structure of the group S_n is known to the reader: e.g. the classes of S_n correspond to the partitions of number n.

2. Subgroup structure. Recall a group G with no non-trivial normal subgroups is called *simple*. Finite simple groups are the *atoms* in the ($\mathcal{G}^{\circ\circ}$) category of finite groups, meaning any finite group is an extension (composition) of more elementary, simple groups; for example, Sym_3 is composed of Z_3 and Z_2, and Alt_4 is also, as we shall see that $Alt_4 = (Z_2 \times Z_2) \rtimes Z_3$).

   For any finite group G, the *Hasse diagram* exhibits the subgroup structure; for example, for Z_6 the Hasse diagram is



(A.4)

   Really, the set of subgroups H of a group G form a *lattice* (we do not elaborate; see e.g. [25]).

   Jordan-Hölder theorem. For any group G, a *normal* subgroup H is maximal, if there is NO group K, with $H \subset K$ and K normal in G. Then $G/H = Q_{u_1}$ is simple (proof elemental!); if now H is not simple, it has Q_{u_2} as quotient H/K, with K maximal in H. So we obtain a chain: $\{G; Q_{u_1}; Q_{u_2}; ...Q_{u_n}\}$, or $\{G; H; K_1; ...K_n\}$ with Q_j simple. The chain ends when the last K_n is already *simple*. The Jordan-Hölder *theorem* asserts: any other choice of maximal normal subgroups has the same chain of quotients, up to the order. See e.g. [17], [25], etc... By contrast, a group G is called *solvable* if the chain of Jordan-Hölder quotients is made out of abelian (hence prime cyclic) groups.

   When we have an extension $Z_n \to G \to Q$, we say that $G = n \cdot Q$, and called it a Schur *multiplier*. See Sect. 12.

3. Abelian Groups. The category of finite abelian groups is perfectly understood (Frobenius and Stickelberger, 1879); see any textbook or even [7].

   If A is abelian ($A = G_{ab}$), one usually writes 0 for e and sum $(a+b)$ for composition; Z_n is abelian for any n, and so are $Sym_2(\approx Z_2)$ and $Alt_3(\approx Z_3)$. As any *irrep* is 1-d (see App. B), the Character Table conveys the whole *irrep* information.

   If A is (finite and) abelian, it is a direct product (direct sum) of cyclic groups of order power of a prime, $A = P(p)Q(q)...T(t)$, where $|P(p)| = p^f$, etc., with $p, q, ...t$ primes. If α is an automorphism of $A = P(p)Q(q)$, and p', q' are in each factor, any *auto* α preserves the order, hence also the product structure, that is $\alpha(p') = p'' \in P$, $\alpha(q') = q'' \in Q$.

For rank r of an abelian group we understand the *minimal* number of generators; it does *not* coincide with the number of different number of prime-power products: for example, Z_6 is rank one, but it is also $Z_2 \times Z_3$.

For $|A| = p^f$, there are as many different abelian groups of this order as *partitions* of number f, called $Part(f)$, that is expressions of f as sum of integers; we have e.g.,

$$Part(1,2,3,4,5,6,7,8,9,10) = 1,2,3,5,7,11,15,22,30,42. \tag{A.5}$$

For example, the number of abelian groups of order $32 = 2^5$ is $Part(5) = 7$, namely $Z_{32}, Z_{16} \times Z_2, Z_8 \times Z_4, Z_8 \times V, {Z_4}^2 \times Z_2, Z_4 \times {Z_2}^3$ and ${Z_2}^5$; see e.g. ([25], Ch. 4). The number of *abelian* groups of order 720 is ten: $720 = 2^4 \cdot 3^2 \cdot 5$, and so $Part(4) \cdot Part(2) \cdot Part(1) = 5 \cdot 2 \cdot 1$. $Part(n)$ grows rather quickly with n, like [23]

$$Part(n) \approx \exp\left(\frac{\pi\sqrt{2n/3}}{4n\sqrt{3}}\right), \quad n \gg 1. \tag{A.6}$$

We shall need and use some results on $Aut(A)$, A abelian; the complete results are messy; for a recent review, see [19].

For any (abelian or not) group G one can form the holomorph, $Hol(G)$, defined by

$$Hol(G) = G \rtimes Aut(G). \tag{A.7}$$

For example, $Hol(Z_3) = S_3 (= Z_3 \times Z_2)$; we shall use the holomorph concept mainly for *abelian* groups. Finally we recall (Appendix B) the *irreps of an abelian group are unidimensional, and reciprocally.*

4. *G operates in Ω: Actions of Groups. The group $G = \{g\}$ operates in the set $\Omega = \{x\}$, if $g(x) = y$ is well defined, with $e(x) = x$ and $g(k(x)) := (gk)(x)$; for this action* we shall write at times $G \circ\!\!\rightarrow \Omega$. This is equivalent to the existence of a natural map

$$\mu : G \to Perm(\Omega), \tag{A.8}$$

where $Perm(\Omega) = Sym_n$ if $|\Omega| = n$: $Perm(\Omega)$ is kind of "universal" operating group in the set Ω. The true nature, its *raison d'etre*, of groups is as transformation groups in some sets: the idea of *Geometry* (F. Klein) is just that of a group of allowed transformations G in some space Ω [21]. Also, *Symmetries* in physics are implemented by means of groups.

If the kernel above, $K := ker\,\mu$ is $= e$, we say the action (of G on Ω) is *effective*. G/K acts naturally and *effectively* on Ω. Call $G(x)$ or $G \cdot x$ the set of y's in Ω reached from

x by some $g \in G$: it is called the *orbit* of x; *belonging to an orbit* is an equivalence relation. If in $G \circ\!\!\rightarrow \Omega$ there is only an orbit, we speak of action being *transitive*: any element x can go to any other y. In any orbit, the set $G_x = \{g \in G | g(x) = x\}$ is a subgroup of G, called the *stabilizer* of point x (*isotropy* subgroup, *little group* in physics), or of the orbit $G(x)$: different points in the same orbit have *conjugate* stabilizers; in our finite group context, clearly $|Orb\, x| = |G|/|G_x|$, or

$$|G| = |Orb\, x| \cdot |G_x| \quad \text{for any } x. \tag{A.9}$$

Orbits with a single point signal *fixed* points, whose stabilizer is the whole group; for example, $GL_n(\mathbb{R})$ acting on vector space $\mathbb{R}^n$ has the origin $x = 0$ as (unique) fixed point. Or: central elements of a group are the fixed points for the adjoint action (conjugation).

*For example, $SO(3)$ acting by rotations is *transitive* in the ordinary sphere S^2 ; the stabilizer (of any point, e.g. the north pole) is $\approx SO(2)$*. If T_2 is the regular triangle $\triangle$, the symmetric group S_3 acts by orthogonal transformations (leaving it invariant), $S_3 \circ\!\!\rightarrow T_2$: the action is transitive, with stabilizer Z_2 as reflections. The subgroup $Alt_3 = Z_3$ of rotations is also transitive, with e as stabilizer: one says then the action is *free*. If G with $|G| = n$ operates in X, $cardX = m$ and describes k orbits, one has $|G| = |G_x| \cdot |G(x)|$, for x arbitrary in each of the k orbits as in A.9.

The action $G \circ\!\!\rightarrow X$ is *2-transitive*, if it is transitive and G_x acts transitive in the complement to x, $X^* := X \backslash \{x\}$, i.e., the little group acts still *trans* in the complementary. Alternatively, if any two points $x \neq y$ go to any other two points $g(x) \neq g(y)$. One defines *k-transitive* actions of G on X, by the natural generalization: in particular, G operating on X is k-transitive, if any k different points go to any other set of distinct k points. In particular, by a transitive action we just mean 1-transitive.

The k-transitive action is called *sharp*, if the last stabilizer is just the identity: in particular, if G is sharp 1-trans on Ω, $|\Omega| = |G|$; if $G \circ\!\!\rightarrow \Omega$ is 2-*trans* sharp, we have similarly $|G| = |\Omega| \cdot (|\Omega| - 1)$, etc. Nearly by definition, $Sym_n = S_n$ is sharp n-transitive in n objects; one shows easily that Alt_n is $(n-2)$ sharp transitive in n objects (as $Alt_3 = Z_3$). Besides the natural Sym_n and Alt_n, actions more than 3-transitive are very rare; but with finite fields $\mathbb{F}_q$ (App. B), one shows that $PGL_2(q)$ is sharp 3-transitive in the projective line $\mathbb{F}_q P^1$, with $q+1$ points, where $q = p^f$ (power of a prime). *Compare $\mathbb{R}P^1 \approx S^1$, where $x \to (ax+b)/(cx+d)$ is sharp 3-*trans*, with stabilizers $Aff_1(\mathbb{R}), \mathbb{R}\backslash\{0\}, \{e\}$*.

5. Examples of families of finite groups. We already defined and use

$$Z_n(\text{order } n),\ D_n(\text{oder } 2n),\ Sym_n = S_n(\text{order } n!),\ Alt_n(n!/2) \tag{A.10}$$

as families of finite groups; some structural relations are

$$Z_p \text{ is simple; } D_n = Z_n \rtimes Z_2; \; Alt_3 = Z_3; \; Sym_2 = Z_2;$$
$$Sym_3 = D_3; \; Alt_4 = V \rtimes Z_2; \; S_4 = V \rtimes S_3 = Hol(V). \tag{A.11}$$

Dicyclic groups are defined as 2-quotients, namely (If a, α verify $a^{2n} = \alpha^4 = e$),

$$(Dicl_n =) Q_n := Z_{2n} \rtimes\!/_2 \, Z_4, \quad \alpha a \alpha^3 = a^{-1}, \quad a^n = \alpha^2. \tag{A.12}$$

We have

$$|Q_n| = 4n, \qquad Q_2 = Q(= \pm(1, i, j, ij = k), \qquad Q_3 \approx Z_3 \rtimes Z_4, \qquad etc. \tag{A.13}$$

where Z_4 acts in Z_{2n} by the inverse auto via the $2:1$ map $Z_4 \to Z_2$.

We shall define another family, Γ_n, the finite Clifford groups [6]: define Dirac-like matrices γ_μ by

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}; \quad \mu, \nu = 1, 2, ..., n. \tag{A.14}$$

The γ's make up a *finite* group of order 2^{n+1}:

$$\pm\{1; \gamma_\mu; \gamma_\mu\gamma_\nu; ...; \text{“}\gamma_5\text{”} := \gamma_1\gamma_2...\gamma_n\}. \tag{A.15}$$

One has $\Gamma_1 = Z_4$, $\Gamma_2 = Q$, Γ_4 is the group of the usual Dirac matrices (order 32), etc. The even order matrices make up a index-2 subgroup, hence normal, called ${\Gamma_n}^+$; for example ${\Gamma_2}^+ = Z_2$.

B APPENDIX

1A.- A representation of the group G is a natural map (morphism) $D : G \to Aut(V)$, where V is a K-vector space; we shall consider only spaces V *finite* dimensional over $\mathbb{C}$, the field of *complex* numbers (at times over $\mathbb{R} \subset \mathbb{C}$): so a representation of a group is a realization of the group via complex (invertible) matrices; of course, $Aut(V) = GL_n(K)$, if $dim\, V = n$: $GL_n(K)$ is the set of $n \times n$ invertible matrices with entries in the field K. Representation theory was instrumental in extending group theory at the end of the XIX century (Frobenius, 1896; Schur...), and it has many applications in physics, mainly in Quantum Physics [30], [31].

If a subspace $W \subset V$ is invariant, namely if $D(G)W \subset W$, we say D *reduces* to W; D is irreducible if only V and $\mathbb{C}$ are invariant. A representation D of group G is called fully reducible or *decomposable*, if for any subrepresentation D_1 in $W \subset V$ there is a complementary one D_1' in W', so one writes $D = D_1 + D_1'$ and $V = W + W'$. Finite groups (indeed, compact groups) have *only* decomposable representations, as one can set up a metric in V, the representation space, so D is unitary under that metric: then any invariant subspace has an orthogonal one, also invariant.

$D : G \to Aut(V)$ and $D' : G \to Aut(V')$ are equivalent, if $\exists A : V \Leftrightarrow V'$ with $D'(g) = AD(g)A^{-1}$. The search for *irreducible inequivalent representations (irreps)* is a formidable industry, with plenty of applications in mathematics and in physics. For a group G, note the set of irreps by $\hat{G}$: it is a well-defined set.

We have, for any finite group G, the following results ([25], [30]):

1) The number of *irreps* coincides with the number of classes, r.

Hint: The formal sums $\sum x_i g_i$ with x_i complex numbers, make up a complex associative algebra $\mathcal{A}(G)$, as the "units" g multiply by the group product, $g_i g_j = g_k$, with dimension of $\mathcal{A} = n \equiv |G|$; this is called the *group algebra* $\mathcal{A}(G) = \mathcal{A}_{\mathbb{C}}(G)$ (Frobenius, 1906). The center of this algebra is generated by $g(= ege) + \sum kgk^{-1}$, i.e. by sum of classes of conjugate elements, so the order of the center is r, the number of classes. One obtains a bilateral central projector for each class. One shows then that $\mathcal{A}(G)$ splits in m simple matrix algebras $\mathcal{M}_i$, where $m = dim(\text{center}\mathcal{A}) = \#$ of classes $= r$. It is well-known (e.g. [30]) that simple complex matrix algebras are *complete*, and of the form $\mathcal{M}_n(\mathbb{C})$, hence a square; so it follows that $\mathcal{A}(G) = \sum_{i=1}^{r} \mathcal{M}_i{}^2$.

This leads to the important and very much used

2) Burnside relation between the order $|G|$ and the $dim\, d_i$ of the i-th irrep:

$$\boxed{|G| = \sum_{i=1}^{r} d_i^2} \tag{B.1}$$

3) If G abelian, all irreps are of dim 1 (Schur's Lemma; obvious from $cl(a) = \{a\}$); in this case one shows $\hat{A}$ is an *abelian group*, and $A \approx \hat{A\,}\hat{}$: this is the starting point of *Fourier analysis*: for example, if $G = U(1)$, the dual $\hat{U(1)} = Z$, and completeness in either space is justly Fourier series!

4) When $G/K = Q$, the set of *irreps* of G includes these of Q (as $D : G \to Q \to Aut(V)$ defines $D : G \to Aut(V)$).

5) $|G_{ab}|$ is the number of 1-dim irreps of G (always ≥ 1).

6) If $G = K \times H$, $D(G) = D(K) \otimes D(H)$, where $\otimes$ is Kronecker product of matrices.

7) For A abelian, $Aut(A)$ operates in $\hat{A}$. For G arbitrary, it is $Out(G)$ which operates in $\hat{G}$.

8) If G finite, $|G| : d_i$, i.e., the dim of the *irrep* divide G (difficult; see [27]).

1B.- Character Table. For any representation $D : G \circ\!\!\to V$ the character χ of element g in this representation D is defined as the trace: $\chi : G \to \mathbb{C}^*$

$$\chi_D(g) := TrD(g) \tag{B.2}$$

One proves at once: elements in the same class have the same character, and equivalent representations also; both come from the *cyclic* property, namely $Tr(ABC) = Tr(CAB)$.

As an example, for the group Z_3 the Character Table runs as follows: (with ω a cubic root, $\omega = \exp(2\pi i/3)$).

	1A	3A	3B
D_0	1	1	1
D_1	1	ω	ω^2
D_2	1	ω^2	ω

As finite groups are compact, and *irreps* of compact groups are unitary (mentioned above), there are orthogonality properties between rows and between columns [30].

We write only one: the square modulus of the first row is again Burnside relation, namely

$$|G| = \sum_{i=1}^{r} d_i^2. \tag{B.3}$$

Different columns have zero (conjugate) product: above e.g. it is $(1 + \omega + \omega^2 = 0)$, as the sum of the three cubic roots of 1.

2. Finite Fields. Some of our finite groups can be seen as groups of matrices over *finite fields*. Let p be a prime, and in the numbers $0, 1, 2, ..., (p-1)$ consider sum and product *mod p*: the *congruences* of Gauss; for example, in $(0, 1)$ the rules are $0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 0$; with $0 \cdot (any) = 0$, and $1 \cdot 1 = 1$. So we *define* the finite field $\mathbb{F}_2$ with 2 elements, and the finite field $\mathbb{F}_p$ with p; $\mathbb{F}_p^* := \mathbb{F}_p \setminus \{0\}$ is the multiplicative group: only 0 is non-invertible under the product. These finite fields were first introduced by Galois (1832).

   One shows (Moore 1903) that any other finite field $\mathbb{F}_q$ has cardinal $q = p^f$, for (any prime p and) any natural number f, and it is commutative (Wedderburn, 1905); but the rules are *different*:

$$\text{Sum in } Z_q, \text{ as in the elementary abelian group } (Z_p)^f. \tag{B.4}$$

$$\text{Product in } F_q^*, \text{ as in } Z_{q-1}. \tag{B.5}$$

   For example,

$$\text{in } \mathbb{F}_4: \ \{0, 1, \omega, \omega^2\} \ \text{ is e.g. } \ 1 + \omega = \omega^2, \ \omega + \omega^2 = 1. \qquad 1 + \omega^2 = \omega. \tag{B.6}$$

   Product: as Z_3 in $(1, \omega, \omega^2) = (e, a, a^2)$.

   Given any finite field $\mathbb{F}_q$, $\mathbb{F}_q^n$ is the *n-dim* $\mathbb{F}_q$-vector space, $M_n(\mathbb{F}_q)$ all the $n \times n$ matrices $\approx End(\mathbb{F}_q^n)$, and $GL_n(\mathbb{F}_q) = GL_n(q)$ the invertible ones. One shows the order of the *finite* group GL is

$$|GL_n(q)| = (q^n - 1)(q^n - q)(q^n - q^2)...(q^n - q^{n-1}), \tag{B.7}$$

   (e.g. in the first row there cannot be all 0's; the second is linear independent of the first, etc.).

   The determinant map: $GL_n(q) \to \mathbb{F}_q^*$ has as kernel the unimodular group $SL_n(q)$. Also the diagonal matrices $\approx \mathbb{F}_q^*$ are normal (central) in GL, with PGL as quotient.

So one constructs the Box

$$\begin{array}{ccccc} H & \longrightarrow & SL_n(q) & \longrightarrow & PSL_n(q) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{F}_q^* & \longrightarrow & GL_n(q) & \longrightarrow & PGL_n(q) \\ \downarrow & & \downarrow & & \downarrow \\ J & \longrightarrow & \mathbb{F}_q^* & \longrightarrow & H \end{array}$$

where $H = \mathbb{F}_q^* \cap SL_n(q)$, etc. Now, the *fundamental result of Jordan-Dickson says:* $PSL_n(q)$ *is* simple, *for all* n *and* q, *except* $n = 2$ and $q = 2$ or $q = 3$. Indeed

$$PSL_2(2) = Sym_3 = Z_3 \rtimes Z_2; \quad PSL_2(3) = Alt_4 = V \rtimes Z_3. \tag{B.8}$$

This forms the *third* family (besides Z_p and $Alt_{n>4}$) of *finite simple* groups, although it is *bi*-parametric. Other families, that we omit, contain the equivalent to the orthogonal and symplectic groups, plus the corresponding exceptional groups (Chevalley, 1955), etc.

3. $|G| =$ up to pq. Let us study the possible finite group with $|G|$ up to order pq, only two factors. If only one, Z_p is the only possibility, and the group split in *irreps* and classes is

$$Z_p : \{a^p = e\}; \quad p = p \cdot 1^2 \quad \text{and} \quad p = 1 \cdot 1_1(e) + (p-1) \cdot 1_p. \tag{B.9}$$

These groups are finite, abelian, cyclic and simple: one for each prime (e.g. Z_{137}).

Consider now $|G| = p^2$; one has $p^2 = p^2 \cdot 1^2$, so they are abelian; and there are two of them, as $Part(2) = 2$: they correspond to Z_{p^2}, cyclic, and $(Z_p)^2$, the elementary abelian: the groups $(Z_p)^f$, appearing very often, are called *elementary abelian groups*.

E.g. for the case $(Z_p)^2$ it is,

$$Z_p \times Z_p : \quad p^2 = p^2 \cdot 1^2; \quad p = 1 \cdot 1_1(e) + (p-1) \cdot 1_p \quad \text{(rest)}. \tag{B.10}$$

Next case is $|G| = p(\neq)q$; if $p = 2$, there is also the dihedral group, performing the *auto* of pass to the inverse in Z_p: so there are just *two* groups,

$$|G| = 2q : \quad Z_{2q} \quad \text{and} \quad D_q = Z_q \rtimes Z_2. \tag{B.11}$$

We leave to the reader to express $2q =$ (sum of irreps) = (sum of classes).

Next case is $|G| = pq$, $2 < p < q$. Only possible non-abelian solution is

$$pq = p \cdot 1^2 + x \cdot p^2, \quad \text{with} \quad x = (q-1)/p, \tag{B.12}$$

when $(q-1):p.$ we say then that primes q and $p(>2)$ are *compatible*; $q, p = 3, 7$ are the first example (see more examples in [7]); then we have the *semidirect product*

$$pq = |G_{pq}| = p \cdot 1^2 + (q-1)/p \cdot p^2 \quad \text{or} \quad G_{pq} = Z_q \rtimes Z_p. \tag{B.13}$$

When p and q are incompatible, the only possible group is the cyclic Z_{pq}.

We shall not bother to completely classify the pqr groups; only *two* cases will be considered here:

(a) $|G| = p^3$: there are 3 abelian groups, as $Part(3) = 3$, and two nonabelian groups, repeating our studied case for $|G| = 8$ and 27: one has the two constructs

$(Z_{p^2}) \rtimes Z_p$, possible as $|Aut(Z_{p^2})| = \Phi(p^2 - p)$ and $\exists Z_p \to Aut(Z_{p^2})$.
$(Z_p)^2 \rtimes Z_p$, possible also, as $|Aut(Z_p)^2| = |GL_2(p)| = (p^2-1)(p^2-p)$, divides p.

The two nonabelian groups of order p^3 can be seen to be *extraspecial* [20].

(b) $G = (Z_p)^3$, an elementary abelian group. In this case $G = {\mathbb{F}_p}^3$, and $Aut(G) = GL_3(\mathbb{F}_p)$, of order $(p^3-1)(p+1)(p-1)^2p^3$.

4. Coxeter groups. H. S. M. Coxeter (1907 - 2003) studied since 1930 discrete groups generated by involutions (reflections); for full information see [10].

There is a well-known graphical representation (invented by Coxeter, but copied by Dynkin):

(a) ∘ means a single involution a, so the group is Z_2.

(b) ∘ ∘ two independent involutions, so the group is $V = (Z_2)^2$.

(c) ∘—∘ means, by definition, $a^2 = b^2 = (ab)^3 = 1$: it generates Sym_3.

(d) $\circ\overset{4}{—}\circ$ means $a^2 = b^2 = (ab)^4 = e$; it generates D_4. With any other number n it generates the dihedral D_n, order $2n$. It is the isometry group (rotations and reflections) of a regular polygon of n sides.

(e) ∘—∘—∘—...—∘, means $a^2 = b^2 = ... = n^2 = (ab)^3 = (bc)^3 = (ac)^2 = ... = (mn)^3 = e$: it generates S_n (easy to prove).

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